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SEMI DISCRETE FORMULATION OF GALERKIN -PARTIAL ARTIFICIAL DIFFUSION FINITE ELEMENT METHOD FOR COUPLED BURGERS' PROBLEM

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Abstract: The viscous Burgers' equation is an example of an equation that has unstable Galerkin finite element approximation for the convection-dominated case. In this paper, a stabilized finite element method for solving coupled Burgers' equation in 2-D is studied, the method consists of adding artificial viscosity acting only on the small scales. We consider a semi-discrete formulation, the theoretical evidence proved the stability and proved that the error estimate is of order $O(h^{2r})$. The discretization with respect to space variables only is applied, whereas time remains continuous. This leads to a large system of ordinary differential equations which are solved using the MATLAB function ODE45 where the numerical results are compared with the exact solution.

Keywords: Galerkin -Partial Artificial Diffusion, Error Estimate, Burgers' Equation.

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INTRODUCTION

In recent years, Burgers' equation have received a considerable amount of attention due to the large number of physically important phenomena that can be modeled using this equation. Some attention has been given to the convection-dominated case. Several methods have been intensively studied to remove such a drawback for this problem and relevant problem, we can summarize some of the most relevant literature. Atwell and King [1] used the Galerkin and Galerkin Least-Squares approximation for 1-D Burgers' equation with the linear feedback control law designed for the non-stabilized problem. Heitmann[5] applied subgrid scale eddy viscosity for convection dominated diffusive transport. The method consists of adding artificial viscosity term $\alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla v_h)$ of orthogonal projection acting only on the fine scales, he give a comprehensive analysis of this method, in [6] he applied this method in a finite difference method by using an appropriate interpretation of the term $\alpha(P_{L_H}^{\perp}\nabla u_h, \nabla v_h) \equiv \alpha(\nabla u_h, \nabla v_h) - \alpha(\nabla \bar{u}_h, \nabla v_h)$, where $\bar{u}$ is an average over itself and its five nearest discrete neighbors. Volkwein [10] considered upwind techniques and mixed finite elements for the steady-state Burgers' equation in 1-D to compute solutions for small viscosity parameters . Diez, Gunzburger and Kunoth [4] studied the multiscale finite element viscosity method for hyperbolic conservation laws which was applied to 1-D inviscid Burgers' equation developed in terms of hierarchical finite element bases to a wavelet basis for a controlled adaptive resolution of discontinuities of the solution. Chan, Demkowicz, Moser and Roberts [3] applied Discontinuous Petrov-Galerkin to 1-D Burgers' and compressible Navier–Stokes equations, they were able to solve problems with Reynolds number up to Re $=10^{10}$. Kashkool and Noon [7] used the classical artificial diffusion for Galerkin and Galerkin-Conservation finite element methods for coupled Burgers' problem in 2-D for the convection-dominated case. The fully discrete formulation with the backward Euler -Galerkin and Galerkin-Conservation schemes were considered. the error estimate of these methods were of order $O(h, k)$ and the numerical results were compared with the exact solution. In this paper, we present a stabilized finite element method for solving coupled Burgers' problem in 2-D, the method consists of adding artificial viscosity acting only on the fine scales to a variational formulation of the problem. We consider a semi-discrete formulation, we prove the stability and convergence for the approximation. The numerical solution of this method is compared with the exact solution . This paper is organized as follows. In section 2, definitions and some important theorems are given. In Section 3, we present the time-dependent modeling problem and a weak form of 2-D Burgers' problem and we show the continuity and V –elliptic. The semi-discrete approximation, stability and error estimate are presented in section 4. In section 5 the finite element approximation, test problem and numerical results are introduced. The conclusions is shown in section 6.

2- Definitions and some important theorems

It is beneficial to mention the definitions and some important theorems which will be used frequently in the equal .

Definition 2.1 [5] For $\Omega \subset R^m$ the (a, b) weighted norm of a function $u : \Omega \rightarrow R$ is defined by, $\|u\|^2_{a,b} = a\|u\|^2 + b\|\nabla u\|^2$.

Definition 2.2 [5] For $\Omega \subset R^m$ the (a, b, α) weighted norm of a function $u : \Omega \rightarrow R$ is defined by, $\|u\|^2_{a,b,\alpha} = \|u\|^2_{a,b} + \alpha\|P_{L_H^{\perp}} \nabla u\|^2$.

The operator $P_{L_H^{\perp}}$ is a projection operator which will be discussed in section 4.

Definition 2.3 [5] Given any real number c define the weighted norm $\|.\|_{(c:(0,T))}$ on $\Omega \times (0, T]$ to be, $\|\mu\|^2_{(c:(0,T))} = \int_0^T e^{c(t-T)} \|\mu\|^2 dt.$

Theorem 2.1 (Poincaré Inequality). [8] Let $\Omega \subset R^2$ be a bounded domain. Then, there is constant $C = C(\Omega)$, such that for any $u \in H_0^1$, $\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$. (2.1)

Theorem 2.2 (Lax–Milgram Theorem).[8] Let V be a Hilbert space with inner product $(.,.)$ and let $a(.,.)$ be a coercive continuous bilinear form on V , and let $l(.)$ be a continuous linear form on V . Then, there exist a unique solution $u \in V$ to the abstract variational problem: find $u \in V$ such that, $a(u, v) = l(v), \quad \forall v \in V$.

Theorem 2.3 (Inverse Estimate). [8] On a quasi-uniform mesh any $u \in V_h$ satisfies the inverse estimate, $\|\nabla u\|_{L^2(\Omega)} \leq Ch^{-1}\|u\|_{L^2(\Omega)}$. (2.2)

Abstract Results [2]

Let H be a separable Hilbert space , let a be a bounded symmetric bilinear form on H with the property that for some $\delta > 0$, $a(u, u) \geq \delta\|u\|^2, \quad \forall u \in H,$ (2.3)

and let B be a trilinear form on H such that there exists a constant $\beta > 0$ such that

$$|B(u, v, w)| \leq \beta \|u\| \|v\| \|w\|, \quad \forall u, v, w \in H. \quad (2.4)$$

Remark 2.1 [5]**:** We shall state the following inequality which will be used frequently in this paper, $\|u - Iu\| \le Ch^r \| u \|_r$, $1 \le r \le s,\ s \ge 2.$ (2.5)

3- Time-Dependent Modeling Problem

Consider the nonlinear time-dependent for the two dimensional coupled Burgers' problem[11].

$$u_t - \epsilon\Delta u + u\, u_x + v\, u_y = f, \qquad (3.1.a)$$

$$v_t - \epsilon\Delta v + u v_x + v v_y = g, \qquad (3.1.b)$$

with boundary conditions

$$u(x, y, t) = 0 \quad on \quad \partial\Omega \times (0, T], \qquad v(x, y, t) = 0 \quad on \quad \partial\Omega \times (0, T],$$

and initial conditions

$$u(x, y, 0) = u^0(x, y), \quad v(x, y, 0) = v^0(x, y),$$

Where $\epsilon > 0$ is a viscosity constant, $\Omega \subset R^2$ with boundary $\partial\Omega$, $u = u(x, y, t)$, $v = v(x, y, t)$, f and $g \in L^2(\Omega)$.

The weak formulation of (3.1) is : find $u, v \in V = H_0^1(\Omega)$ such that:

$$(u_t, \varphi) + a(u, \varphi) + B(u, u, \varphi) + B(v, u, \varphi) = (f, \varphi), \qquad (3.2.a)$$

$$(v_t, \varphi) + a(v, \varphi) + B(u, v, \varphi) + B(v, v, \varphi) = (g, \varphi), \forall \varphi \in H_0^1(\Omega), \qquad (3.2.b)$$

$$(u(x, y, 0), \varphi) = (u^0, \varphi), \qquad (v(x, y, 0), \varphi) = (v^0, \varphi),$$

Where $a(u, \varphi) = (\epsilon \nabla u, \nabla\varphi)$ and $a(v, \varphi) = (\epsilon \nabla v, \nabla\varphi)$, $B(u, u, \varphi) = (u\, u_x, \varphi)$, $B(v, u, \varphi) = (v u_y, \varphi)$, $B(u, v, \varphi) = (u\, v_x, \varphi)$, $B(v, v, \varphi) = (v\, v_y, \varphi)$,

The weak formulation (3.2) with artificial viscosity term $\alpha(P_{L_H}^{\perp} \nabla u, P_{L_H}^{\perp} \nabla\varphi)$ is : find $u, v \in V = H_0^1(\Omega)$ such that:

$$(u_t, \varphi) + A(u, \varphi) = (f, \varphi), \qquad (3.3.a)$$

$$(v_t, \varphi) + A(v, \varphi) = (g, \varphi), \quad \forall \varphi \in H_0^1(\Omega), \qquad (3.3.b)$$

Where, $A(u,\varphi)=a(u,\varphi)+\alpha(P_{L_H^{\perp}}\nabla u,P_{L_H^{\perp}}\nabla\varphi)+B(u,u,\varphi)+B(v,u,\varphi)$,

$A(v,\varphi)=a(v,\varphi)+\alpha(P_{L_H^{\perp}}\nabla v,P_{L_H^{\perp}}\nabla\varphi)+B(u,v,\varphi)+B(v,v,\varphi)$.

Assumption 3.1:

(A1) There exists a constant μ such that: $\epsilon\geq\mu>0$.

We assume that the solution u and v of problem (3.2) satisfied the following condition

(A2) $u,v\in L^{\infty}(0,T,H_0^1(\Omega))\cap L^{\infty}(0,T,H^2(\Omega))$, $u_t,u_{tt},u_{ttt},v_t,v_{tt},v_{ttt}\in L^{\infty}(0,T,L^{\infty}(\Omega))$.

Lemma 3.1: $A(u,\varphi)$ and $A(v,\varphi)$ given by (3.3) are continuous and V-elliptic(coercive).

Proof: 1-For continuity ,we have,

$|A(u,\varphi)|\leq|\epsilon(\nabla u,\nabla\varphi)|+|\alpha(P_{L_H^{\perp}}\nabla u,P_{L_H^{\perp}}\nabla\varphi)|+|(uu_x,\varphi)|+|(vu_y,\varphi)|$,

$|A(v,\varphi)|\leq|\epsilon(\nabla v,\nabla\varphi)|+|\alpha(P_{L_H^{\perp}}\nabla v,P_{L_H^{\perp}}\nabla\varphi)|+|(uv_x,\varphi)|+|(vv_y,\varphi)|$,

$|A(u,\varphi)|\leq|\epsilon(\nabla u,\nabla\varphi)|+|\alpha(P_{L_H^{\perp}}\nabla u,P_{L_H^{\perp}}\nabla\varphi)|+|(u\nabla u,\varphi)|+|(v\nabla u,\varphi)|$,

$|A(v,\varphi)|\leq|\epsilon(\nabla v,\nabla\varphi)|+|\alpha(P_{L_H^{\perp}}\nabla v,P_{L_H^{\perp}}\nabla\varphi)|+|(u\nabla v,\varphi)|+|(v\nabla v,\varphi)|$,

Applying Cauchy-Schwartz inequality gives,

$|A(u,\varphi)|\leq\epsilon\|\nabla u\|\,\|\nabla\varphi\|+\alpha\|P_{L_H^{\perp}}\nabla u\|\,\|P_{L_H^{\perp}}\nabla\varphi\|+\|u\|\,\|\nabla u\|\,\|\varphi\|+\|v\|\,\|\nabla u\|\,\|\varphi\|$,

$|A(v,\varphi)|\leq\epsilon\|\nabla v\|\,\|\nabla\varphi\|+\alpha\|P_{L_H^{\perp}}\nabla v\|\,\|P_{L_H^{\perp}}\nabla\varphi\|+\|u\|\,\|\nabla v\|\,\|\varphi\|+\|v\|\,\|\nabla v\|\,\|\varphi\|$,

From (2.1) we have,

$|A(u,\varphi)|\leq\epsilon\|\nabla u\|\,\|\nabla\varphi\|+\alpha\|P_{L_H^{\perp}}\nabla u\|\,\|P_{L_H^{\perp}}\nabla\varphi\|+\|u\|\,\|\nabla u\|\,\|\varphi\|+C\|\nabla v\|\,\|\nabla u\|\,\|\varphi\|$,

$|A(v,\varphi)|\leq\epsilon\|\nabla v\|\,\|\nabla\varphi\|+\alpha\|P_{L_H^{\perp}}\nabla v\|\,\|P_{L_H^{\perp}}\nabla\varphi\|+C\|\nabla u\|\,\|\nabla v\|\,\|\varphi\|+\|v\|\,\|\nabla v\|\,\|\varphi\|$.

From (A2),we have $C\|\nabla u\|\leq C\|\nabla u\|_{L^{\infty}(H_0^1(\Omega))}\leq C_m$ and $C\|\nabla v\|\leq C\|\nabla v\|_{L^{\infty}(H_0^1(\Omega))}\leq C_m$ thus,

$$|A(u,\varphi)| \leq \epsilon \|\nabla u\| \, \|\nabla\varphi\| + \alpha \|P_{L_H^\perp} \nabla u\| \, \|P_{L_H^\perp} \nabla\varphi\| + C_m \|u\| \, \|\varphi\| + C_m \|\nabla u\| \, \|\varphi\|,$$

$$|A(v,\varphi)| \leq \epsilon \|\nabla v\| \, \|\nabla\varphi\| + \alpha \|P_{L_H^\perp} \nabla v\| \, \|P_{L_H^\perp} \nabla\varphi\| + C_m \|\nabla v\| \, \|\varphi\| + C_m \|v\| \, \|\varphi\|,$$

$$|A(u,\varphi)| \leq N(\sqrt{\epsilon}\|\nabla u\|, \|u\|, \|\nabla u\|, \sqrt{\alpha}\|P_{L_H^\perp} \nabla u\|).(\sqrt{\epsilon}\|\nabla\varphi\|, \|\varphi\|, \|\varphi\|, \sqrt{\alpha}\|P_{L_H^\perp} \nabla\varphi\|),$$

$$|A(v,\varphi)| \leq N(\sqrt{\epsilon}\|\nabla v\|, \|\nabla v\|, \|v\|, \sqrt{\alpha}\|P_{L_H^\perp} \nabla v\|).(\sqrt{\epsilon}\|\nabla\varphi\|, \|\varphi\|, \|\varphi\|, \sqrt{\alpha}\|P_{L_H^\perp} \nabla\varphi\|),$$

From [5] implies,

$$|A(u,\varphi)| \leq N\left(\sqrt{\epsilon\|\nabla u\|^2 + \|u\|^2 + \|\nabla u\|^2 + \alpha\|P_{L_H^\perp} \nabla u\|^2}\right).$$
$$\left(\sqrt{\epsilon\|\nabla\varphi\|^2 + \|\varphi\|^2 + \|\varphi\|^2 + \alpha\|P_{L_H^\perp} \nabla\varphi\|^2}\right),$$

$$|A(v,\varphi)| \leq N\left(\sqrt{\epsilon\|\nabla v\|^2 + \|\nabla v\|^2 + \|v\|^2 + \alpha\|P_{L_H^\perp} \nabla v\|^2}\right).$$
$$\left(\sqrt{\epsilon\|\nabla\varphi\|^2 + \|\varphi\|^2 + \|\varphi\|^2 + \alpha\|P_{L_H^\perp} \nabla\varphi\|^2}\right),$$

from definition 2.2, we have,

$$|A(u,\varphi)| \leq N \, \|u\|_{1,1,\alpha} \|\varphi\|_{1,\epsilon,\alpha},$$

$$|A(v,\varphi)| \leq N \, \|v\|_{1,1,\alpha} \|\varphi\|_{1,\epsilon,\alpha}.$$

2- For ellipticity , we have,

$$A(u,u) = \epsilon(\nabla u, \nabla u) + \alpha(P_{L_H^\perp} \nabla u, P_{L_H^\perp} \nabla u) + (uu_x, u) + (vu_y, u),$$

$$A(v,v) = \epsilon(\nabla v, \nabla v) + \alpha(P_{L_H^\perp} \nabla v, P_{L_H^\perp} \nabla v) + (u\, v_x, v) + (\, vv_y, v),$$

$$A(u,u) \geq \mu\|\nabla u\|^2 + \alpha\|P_{L_H^\perp} \nabla u\|^2 + \|u\|^2 \|u_x\| + \|v\| \, \|u_y\| \, \|u\|,$$

$$A(v,v) \geq \mu\|\nabla v\|^2 + \alpha\|P_{L_H^\perp} \nabla v\|^2 + \|u\| \, \|v_x\| \, \|v\| + \|v\|^2 \|v_y\|,$$

Since $\|u_x\|, \|v\|$, $\|u\|, \|v_y\|$ are non-negative we get,

$A(u,u) \geq \mu \| \nabla u \|^2 + \alpha \| P_{L_H}^{\perp} \nabla u \|^2 + \| u \|^2 + \| u_y \| \, \| u \|,$

$A(v,v) \geq \mu \| \nabla v \|^2 + \alpha \| P_{L_H}^{\perp} \nabla v \|^2 + \| v \| \, \| v_x \| + \| v \|^2,$

From (2.1), we get, $A(u,u) \geq \mu \| \nabla u \|^2 + \alpha \| P_{L_H}^{\perp} \nabla u \|^2 + \| u \|^2 + \frac{1}{C} \| u \|^2,$

$A(v,v) \geq \mu \| \nabla v \|^2 + \alpha \| P_{L_H}^{\perp} \nabla v \|^2 + \frac{1}{C} \| v \|^2 + \| v \|^2,$

$$A(u,u) \geq M \left\{ \| \nabla u \|^2 + \| u \|^2 + \| u \|^2 \right\} + \alpha \| P_{L_H}^{\perp} \nabla u \|^2,$$

$$A(v,v) \geq M \left\{ \| \nabla v \|^2 + \| v \|^2 + \| v \|^2 \right\} + \alpha \| P_{L_H}^{\perp} \nabla v \|^2,$$

from definition (2.1) and (2.2) we have,

$A(u,u) \geq M \| u \|_{1,1}^2 + \alpha \| P_{L_H}^{\perp} \nabla u \|^2, \; A(v,v) \geq M \| v \|_{1,1}^2 + \alpha \| P_{L_H}^{\perp} \nabla v \|^2,$

$$A(u,u) \geq M \| u \|_{1,1,\alpha}^2, \qquad A(v,v) \geq M \| v \|_{1,1,\alpha}^2.$$

4- The Semi-Discrete Approximation

The discrete standard Galerkin finite element method of (3.2), reads: Find an approximation solution $u_h, v_h \in V_h \subset H_0^1$ such that :

$(u_{h,t}, \varphi_h) + a(u_h, \varphi_h) + B(u_h, u_h, \varphi_h) + B(v_h, u_h, \varphi_h) = (f, \varphi_h),$

$(v_{h,t}, \varphi_h) + (v_h, \varphi_h) + B(u_h, v_h, \varphi_h) + B(v_h, v_h, \varphi_h) = (g, \varphi_h), \; \forall \varphi_h \in V_h .$

4.1-Mathematical Formulation of an Artificial Viscosity Term

It is well known that when $\epsilon < h$, where h is mesh size, the convection term dominates over diffusion and the standard Galerkin finite element method produce an oscillating solution which is not close to exact solution. In the following we analyze an approach stabilizing the approximation through the introduction of an artificial viscosity term which acts only on the fine scales of the finite element mesh. We add and subtract $\alpha(\nabla u_h, \nabla \varphi_h)$and $\alpha(\nabla v_h, \nabla \varphi_h)$ to (3.2.a) and (3.2.b) respectively where $\alpha = \alpha\ (h)$ is a positive constant. This gives,

$(u_t, \varphi) + (\epsilon + \alpha)(\nabla u, \nabla \varphi) - \alpha(\nabla u, \nabla \varphi) + B(u, u, \varphi) + B(v, u, \varphi) = (f, \varphi),$

$(v_t,\varphi)+(\epsilon+\alpha)(\nabla v,\nabla\varphi)-\alpha(\nabla v,\nabla\varphi)+B(u,v,\varphi)+B(v,v,\varphi)=(g,\varphi),\ \forall\ \varphi\in H_0^1(\Omega).$

This suggests a mixed methods formulation wherein we define $q_1\equiv\nabla u$ and $q_2\equiv\nabla v\in(L^2(\Omega))^2$ [5]. We obtain the system, find $((u,v),(q_1,q_2))\in(H_0^1,(L^2(\Omega))^2)$ satisfying

$(u_t,\varphi)+(\epsilon+\alpha)(\nabla u,\nabla\varphi)-\alpha(q_1,\nabla\varphi_h)+B(u,u,\varphi)+B(v,u,\varphi)=(f,\varphi),$

$(v_t,\varphi)+(\epsilon+\alpha)(\nabla v,\nabla\varphi)-\alpha(q_2,\nabla\varphi)+B(u,v,\varphi)+B(v,v,\varphi)=(g,\varphi),$

$(q_1-\nabla u,l)=0,\ (q_2-\nabla v,l)=0,\ \forall\ \varphi\in H_0^1(\Omega), l\in(L^2(\Omega))^2.$

In the discretized problem, let h and H represent two mesh widths (with $h<H$). Let $L_H\subset(L^2(\Omega))^2$ and $V_h\subset H_0^1(\Omega)$ be finite element spaces. The problem then is to find $((u_h,v_h),(q_{1H},q_{2H}))\in(V_h,L_H)$ satisfying

$(u_{h,t},\varphi_h)+(\epsilon+\alpha)(\nabla u_h,\nabla\varphi_h)-\alpha(q_{1H},\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(f,\varphi_h),$ (4.1.a)

$(v_{h,t},\varphi_h)+(\epsilon+\alpha)(\nabla v_h,\nabla\varphi_h)-\alpha(q_{2H},\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(g,\varphi_h),$ (4.1.b)

$(q_{1H}-\nabla u_h,l_H)=0, (q_{2H}-\nabla v_h,l_H)=0, \forall\varphi_h\in V_h, l_H\in L_H.$ (4.1.c)

We note that, If $L_H=\{0\}$, L_H is small, then $q_{1H},q_{2H}=0$, and we have a straight artificial viscosity formulation, if L^Hguided by numerical analysis so to obtain a beneficial balance, we set $q_{1H}=P_{L_H}\nabla u_h$ and $q_{2H}=P_{L_H}\nabla v_h$ [5]. Where P_{L_H} is the orthogonal projection of L^2 onto L_H and $P_{L_H^{\perp}}=(I-P_{L_H})$ is orthogonal projection of L^2 on $L_H^{\perp}$, where $L_H^{\perp}=\{w\in L^2(\Omega),(w,s)=0,\ \forall s\in L_H\}$. Then we have $\nabla u_h=P_{L_H^{\perp}}\nabla u_h+P_{L_H}\nabla u_h$ and $\nabla v_h=P_{L_H^{\perp}}\nabla v_h+P_{L_H}\nabla v_h$.

Lemma 4.1 If $q_{1H}=P_{L_H}\nabla u_h$ and $q_{2H}=P_{L_H}\nabla v_h$ then the system (4.1) is equivalent to:

$(u_{h,t},\varphi_h)+a(u_h,\varphi_h)+\alpha(P_{L_H^{\perp}}\nabla u_h,P_{L_H^{\perp}}\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi)=(f,\varphi_h),$

$(v_{h,t},\varphi_h)+a(v_h,\varphi_h)+\alpha(P_{L_H^{\perp}}\nabla v_h,P_{L_H^{\perp}}\nabla\varphi_h)+B(u_h,v_h,\varphi_h)+B(v_h,v_h,\varphi_h)=(g,\varphi_h).$

Proof *:* Using decomposition and L^2 orthogonality, the equation (4.1.c) is satisfied as

$$\begin{aligned}(q_{1H}-\nabla u_h,l_H)&=(P_{L_H}\nabla u_h-\nabla u_h,l_H)\\&=(P_{L_H}\nabla u_h-(P_{L_H}\nabla u_h+P_{L_H^{\perp}}\nabla u_h),l_H)\\&=(P_{L_H^{\perp}}\nabla u_h,l_H)\end{aligned}$$

= 0 (from the definition of $^{\perp}_{L_H}$),

Similarly we get, $(q_{2H} - \nabla v_h, l_H)$=0. For the equation(4.1.a), we get,

$$(\epsilon+\alpha)(\nabla u_h, \nabla\varphi_h)=\epsilon(\nabla u_h, \nabla\varphi_h)+\alpha(P_{L_H}\nabla u_h, \nabla\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla u_h, \nabla\varphi_h)$$

$$=\epsilon(\nabla u_h, \nabla\varphi_h)+\alpha(q_{1H}, \nabla\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}\nabla\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla\varphi_h)$$

$$= \epsilon(\nabla u_h, \nabla\varphi_h)+\alpha(q_{1H}, \nabla\varphi_h)+\alpha\left(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla\varphi_h\right),$$

also for the equation(4.1.b), we get,

$$(\epsilon+\alpha)(\nabla v_h, \nabla\varphi_h)=\epsilon(\nabla v_h, \nabla\varphi_h)+\alpha(q_{2H}, \nabla\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla v_h, P_{L_H}^{\perp}\nabla\varphi_h),$$

Substituting this into (4.1), we have Galerkin with a partial artificial diffusion finite element method which are denoted by (G.P.A.D.),

$$(u_{h,t},\varphi_h)+a(u_h,\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi)=(f,\varphi_h), \quad (4.2.a)$$

$$(v_{h,t},\varphi_h)+a(v_h,\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla v_h, P_{L_H}^{\perp}\nabla\varphi_h)+B(u_h,v_h,\varphi_h)+B(v_h,v_h,\varphi_h) = (g,\varphi_h), \quad (4.2.b)$$

Lemma 4.2 The method described by (4.2) is stable over finite time. specifically, there exist a constant $C > 0$ such that,

$$\|u_h(T)\|^2+\alpha\|P_{L_H}^{\perp}\nabla u_h\|^2_{(\delta:(0,T))} \leq e^{-\delta T}\|u_h^0\|^2+\frac{1}{\delta}\|f\|^2_{(\delta:(0,T))},$$

$$\|v_h(T)\|^2+\alpha\|P_{L_H}^{\perp}\nabla v_h\|^2_{(\delta:(0,T))} \leq e^{-CT}\|v_h^0\|^2+\frac{1}{\delta}\|g\|^2_{(\delta:(0,T))}.$$

Proof *:* Choosing φ_h=u_h in (4.2.a) and φ_h=v_h in (4.2.b) gives,

$$(u_{h,t},u_h)+a(u_h,u_h)+\alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla u_h)+B(u_h,u_h,u_h)+B(v_h,u_h,u_h) = (f, u_h),$$

$$(v_{h,t},v_h)+a(v_h,v_h)+\alpha(P_{L_H}^{\perp}\nabla v_h, P_{L_H}^{\perp}\nabla v_h)+ B(u_h,v_h,v_h)+B(v_h,v_h,v_h) = (g, v_h),$$

Note that, $(u_{h,t},u_h)=\frac{1}{2}\int_\Omega \frac{d}{dt}u_h^2 dx\, dy = \frac{1}{2}\frac{d}{dt}\|u_h\|^2$, $(v_{h,t},v_h) =\frac{1}{2}\int_\Omega \frac{d}{dt}v_h^2\, dx\, dy=\frac{1}{2}\frac{d}{dt}\|v_h\|^2$, From(2.3) we have, $a(u_h,u_h) \geq \delta \|u_h\|^2$, $a(v_h,v_h) \geq \delta \|v_h\|^2$. From(2.4) we get, $B(u_h,u_h,u_h) \leq \beta \|u_h\|^2 \|u_h\|$, $B(v_h,u_h,u_h) \leq \beta \|u_h\|^2 \|v_h\|$,

$B(u_h,v_h,v_h) \leq \beta \|v_h\|^2 \|u_h\|$, $B(v_h,v_h,v_h) \leq \beta \|v_h\|^2 \|v_h\|$.

By using Cauchy-Schwartz inequality and Young's inequality for the right hand sides we have,

$(f,u_h) \leq \|f\| \|u_h\| \leq \frac{1}{2\delta}\|f\|^2 + \frac{\delta}{2}\|u_h\|^2$, $(g,u_h) \leq \|g\| \|v_h\| \leq \frac{1}{2\delta}\|g\|^2 + \frac{\delta}{2}\|v_h\|^2$.

Then,

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \delta\|u_h\|^2 + \alpha\|P_{L_H^\perp}\nabla u_h\|^2 + \beta\|u_h\|^2\|u_h\| + \beta\|u_h\|^2\|v_h\| \leq \frac{1}{2\delta}\|f\|^2 + \frac{\delta}{2}\|u_h\|^2$$

$$\frac{1}{2}\frac{d}{dt}\|v_h\|^2 + \delta\|v_h\|^2 + \alpha\|P_{L_H^\perp}\nabla v_h\|^2 + \beta\|v_h\|^2\|u_h\| + \beta\|v_h\|^2\|v_h\| \leq \frac{1}{2\delta}\|g\|^2 + \frac{\delta}{2}\|u_h\|^2$$

Multiplying by two and rearranging gives,

$$\frac{d}{dt}\|u_h\|^2 + \delta\|u_h\|^2 + 2\alpha\|P_{L_H^\perp}\nabla u_h\|^2 + 2\beta\|u_h\|^2 \|u_h\| + 2\beta\|u_h\|^2 \|v_h\| \leq \frac{1}{\delta}\|f\|^2,$$

$$\frac{d}{dt}\|v_h\|^2 + \delta\|u_h\|^2 + 2\alpha\|P_{L_H^\perp}\nabla v_h\|^2 + 2\beta\|v_h\|^2 \|u_h\| + 2\beta\|v_h\|^2 \|v_h\| \leq \frac{1}{\delta}\|g\|^2.$$

Since, $\beta\|u_h\|^2 \|u_h\|$, $\beta\|u_h\|^2 \|v_h\|$, $\beta\|v_h\|^2 \|u_h\|$ and $\beta\|v_h\|^2 \|v_h\|$ are non-negative, we have, $\frac{d}{dt}\|u_h\|^2 + \delta\|u_h\|^2 + 2\alpha\|P_{L_H^\perp}\nabla u_h\|^2 \leq \frac{1}{\delta}\|f\|^2$,

$$\frac{d}{dt}\|v_h\|^2 + \delta\|v_h\|^2 + 2\alpha\|P_{L_H^\perp}\nabla v_h\|^2 \leq \frac{1}{\delta}\|g\|^2.$$

Multiplying by the integrating factor $e^{\delta t}$ and integrating from $t = 0$ to $t = T$ gives,

$$e^{\delta T}\|u_h(T)\|^2 - \|u_h^0\|^2 + 2\alpha\int_0^T e^{\delta t}\|P_{L_H^\perp}\nabla u_h\|^2 dt \leq \frac{1}{\delta}\int_0^T e^{\delta t}\|f\|^2 dt,$$

$$e^{\delta T}\|v_h(T)\|^2 - \|v_h^0\|^2 + 2\alpha\int_0^T e^{\delta t}\|P_{L_H^\perp}\nabla v_h\|^2 dt \leq \frac{1}{\delta}\int_0^T e^{\delta t}\|g\|^2 dt,$$

this implies, $e^{\delta T}\|u_h(T)\|^2 + \alpha\int_0^T e^{\delta t}\|P_{L_H^\perp}\nabla u_h\|^2 dt \leq \frac{1}{\delta}\int_0^T e^{\delta t}\|f\|^2 dt + \|u_h^0\|^2$,

$$e^{\delta T}\|v_h(T)\|^2 + \alpha\int_0^T e^{\delta t}\|P_{L_H^\perp}\nabla v_h\|^2 dt \leq \frac{1}{\delta}\int_0^T e^{\delta t}\|g\|^2 dt + \|v_h^0\|^2,$$

$$\left\| u_h(T) \right\|^2 + \alpha \int_0^T e^{\delta(t-T)} \left\| P_{L_H}^{\perp} \nabla u_h \right\|^2 dt \le \frac{1}{\delta} \int_0^T e^{\delta(t-T)} \left\| f \right\|^2 dt + e^{-\delta T} \left\| u_h^0 \right\|^2,$$

$$\left\| v_h(T) \right\|^2 + \alpha \int_0^T e^{\delta(t-T)} \left\| P_{L_H}^{\perp} \nabla v_h \right\|^2 dt \le \frac{1}{\delta} \int_0^T e^{\delta(t-T)} \left\| g \right\|^2 dt + e^{-\delta T} \left\| v_h^0 \right\|^2.$$

Using definition (2.3) the proof complete.

For the error analysis we first need to establish the existence of the equilibrium projection $pu_h, pv_h \in V_h$ of u and v respectively which is given by,

$$A(u - pu_h, \varphi_h) = a(u - pu_h, \varphi_h) + \alpha(P_{L_H}^{\perp} \nabla (u - pu_h), P_{L_H}^{\perp} \nabla\varphi_h) + B(u,u,\varphi_h) - B(pu_h, pu_h, \varphi_h) + B(u,u,\varphi_h) - B(pu_h, pu_h, \varphi_h) = 0, \quad (4.3.a)$$

$$A(v - pv_h, \varphi_h) = a(v - pv_h, \varphi_h) + \alpha(P_{L_H}^{\perp} \nabla(v - pv_h), P_{L_H}^{\perp} \nabla\varphi_h) + B(u,v,\varphi_h) - B(pu_h, pv_h, \varphi_h) + B(v,v,\varphi_h) - B(pv_h, pv_h, \varphi_h) = 0 \quad \forall \varphi_h \in V_h. \quad (4.3.b)$$

Lemma 4.3 Let $u, v \in H_0^1(\Omega)$, the equilibrium projection pu_h, pv_h of u, v respectively, given by (4.3) exist uniquely.

Proof: we define for any $u, v \in H_0^1(\Omega)$, $F(\varphi) \equiv A(u,\varphi)$, $G(\varphi) \equiv A(v,\varphi)$, with these definitions we can rewrite (4.3) as, $A(pu_h, \varphi_h) = F(\varphi_h)$, $A(pv_h, \varphi_h) = G(\varphi_h)$.

$F(\varphi_h), G(\varphi_h)$ is a continuous linear functional where lemma 3.1 implies, $|F(\varphi_h)| = |A(u,\varphi_h)|$, $|G(\varphi_h)| = |A(v,\varphi_h)|$. Also since all norms in $H_0^1(\Omega)$ are equivalent in the finite dimensional space V_h, thus from lemma 3.1, $A(pu_h,\varphi_h)$, $A(pv_h,\varphi_h)$ is continuous and coercive, so the hypotheses of Lax-Milgram theorem are satisfied.

Lemma 4.4 Let $u, v \in H_0^1(\Omega)$, let $pu_h, pv_h \in V_h$ be the equilibrium projection given by (4.3) the assumptions of the finite element space there exists a constant C_3 and C_4 independent of ϵ, α, h and H such that

$$\left\| u - pu_h \right\|_{L^\infty(L^2)} \le C_3 (h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}), \quad \left\| v - pv_h \right\|_{L^\infty(L^2)} \le C_4 (h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}).$$

Proof: Let $\eta_1 = u - Iu$, $\eta_2 = v - Iv$, $\vartheta_1 = pu_h - Iu$ and $\vartheta_2 = pv_h - Iv$ where Iu, Iv are the interpolant of u and v in the space V_h. By the triangle inequality we have,

$$\left\| u - pu_h \right\|_{L^\infty(L^2)} \le \left\| \eta_1 \right\|_{L^\infty(L^2)} + \left\| \vartheta_1 \right\|_{L^\infty(L^2)}, \quad \left\| v - pv_h \right\|_{L^\infty(L^2)} \le \left\| \eta_2 \right\|_{L^\infty(L^2)} + \left\| \vartheta_2 \right\|_{L^\infty(L^2)},$$

For the first terms we have, $\|\eta_1\|_{L^\infty(L^2)}$ = $\max_{0\leq t\leq T}\|u - Iu\|$, from (2.5) we have,

$$\|\eta_1\|_{L^\infty(L^2)} \leq \max_{0\leq t\leq T} Ch^r\|u\|_r \leq Ch^r\|u\|_{L^\infty(H^r)}$$

$$\leq C_1 h^r \qquad (4.4.a)$$

also $\|\eta_2\|_{L^\infty(L^2)} \leq Ch^r\|v\|_{L^\infty(H^r)} \leq C_2 h^r$. (4.4.b)

To bound the ϑ_1 and ϑ_2 terms we rewrite equation (4.3)

$$A(pu_h, \varphi_h) = A(u, \varphi_h), \qquad (4.5.a)$$

$$A(pv_h, \varphi_h) = A(v, \varphi_h), \quad \forall\, \varphi_h \in V_h \qquad (4.5.b)$$

subtracting $A(Iu, \varphi_h)$ from (4.5.a) and $A(Iv, \varphi_h)$ from (4.5.b), choosing $\varphi_h = \vartheta_1$ and $\varphi_h = \vartheta_2$ respectively give,

$$a(\vartheta_1,\vartheta_1) + \alpha(P_{L_H}^{\perp}\nabla\vartheta_1, P_{L_H}^{\perp}\nabla\vartheta_1) + B(pu_h,pu_h,\vartheta_1) - B(Iu,Iu,\vartheta_1) + B(pv_h,pu_h,\vartheta_1) - B(Iv,Iu,\vartheta_1) = a(\eta_1,\vartheta_1) + \alpha(P_{L_H}^{\perp}\nabla\eta_1, P_{L_H}^{\perp}\nabla\vartheta_1) + B(u,u,\vartheta_1) - B(Iu,Iu,\vartheta_1) + B(v,u,\vartheta_1) - B(Iv,Iu,\vartheta_1), \qquad (4.6.a)$$

$$a(\vartheta_2,\vartheta_2) + \alpha(P_{L_H}^{\perp}\nabla\vartheta_2, P_{L_H}^{\perp}\nabla\vartheta_2) + B(pu_h,pv_h,\vartheta_2) - B(Iu,Iv,\vartheta_2) + B(pv_h,pv_h,\vartheta_2) - B(Iv,Iv,\vartheta_2) = a(\eta_2,\vartheta_2) + \alpha(P_{L_H}^{\perp}\nabla\eta_2, P_{L_H}^{\perp}\nabla\vartheta_2) + B(u,v,\vartheta_2) - B(Iu,Iv,\vartheta_2) + B(v,v,\vartheta_2) - B(Iv,Iv,\vartheta_2), \qquad (4.6.b)$$

Consider the left hand side of (4.6.a), from (2.3) we have, $a(\vartheta_1,\vartheta_1) \geq \delta\|\vartheta_1\|^2$.

From(2.4) we have, $B(pu_h,pu_h,\vartheta_1) - B(Iu,Iu,\vartheta_1) \leq \beta\left(\|pu_h\|^2 - \|Iu\|^2\right)\|\vartheta_1\| \leq \beta\|pu - Iu\|^2\|\vartheta_1\| = \beta\|\vartheta_1\|^2\|\vartheta_1\|$,

$B(pv_h,pu_h,\vartheta_1) - B(Iv,Iu,\vartheta_1) \leq \beta(\|pv_h\|\,\|pu_h\| - \|Iv\|\,\|Iu\|)\|\vartheta_1\|$.

Similarly for (4.6.b) we get, $a(\vartheta_2,\vartheta_2) \geq \delta\|\vartheta_2\|^2$,

$B(pu_h,pv_h,\vartheta_2) - B(Iu,Iv,\vartheta_2) \leq \beta(\|pv_h\|\,\|pu_h\| - \|Iv\|\,\|Iu\|)\|\vartheta_2\|$,

$B(pv_h,pv_h,\vartheta_2) - B(Iv,Iv,\vartheta_2) \leq \beta\|\vartheta_2\|^2\|\vartheta_2\|$,

For the right hand side of (4.6.a), By using Cauchy-Schwartz inequality we have,

$a(\eta_1,\vartheta_1) \leq \epsilon\|\nabla\eta_1\|\,\|\nabla\vartheta_1\|$, Using the (2.2) on $\|\nabla\vartheta_1\|$ gets, $a(\eta_1,\vartheta_1) \leq Ch^{-1}\epsilon\|\nabla\eta_1\|\,\|\vartheta_1\|$, using Young's inequality gives, $a(\eta_1,\vartheta_1) \leq \frac{3\,C^2h^{-2}\epsilon^2}{2\delta}\|\nabla\eta_1\|^2+\frac{\delta}{6}\|\vartheta_1\|^2$.

From (2.4) we have, $B(u,u,\vartheta_1)-B(Iu,Iu,\vartheta_1) \leq \beta(\|u\|^2-\|Iu\|^2)\|\vartheta_1\|$

$$\leq \beta\|u-Iu\|^2\|\vartheta_1\| = \beta\|\eta_1\|^2\|\vartheta_1\|$$

$$\leq \frac{3\beta^2}{2\delta}\|\eta_1\|^4+\frac{\delta}{6}\|\vartheta_1\|^2.$$

$B(v,u,\vartheta_1)-B(Iv,Iu,\vartheta_1) \leq \beta(\|v\|\,\|u\|-\|Iv\|\,\|Iu\|)\|\vartheta_1\|$, using Young's inequality for the first term gives,

$$B(v,u,\vartheta_1)-B(Iv,Iu,\vartheta_1) \leq \frac{3\beta^2}{2\delta}\|v\|^2\,\|u\|^2+\frac{\delta}{6}\|\vartheta_1\|^2-\beta\|Iv\|\,\|Iu\|\,\|\vartheta_1\|.$$

Similarly for the right hand side of (4.6.b) we get, $a(\eta_2,\vartheta_2) \leq \frac{3\,C^2h^{-2}\epsilon^2}{2\delta}\|\nabla\eta_2\|^2+\frac{\delta}{6}\|\vartheta_2\|^2$.

$$B(u,v,\vartheta_2)-B(Iu,Iv,\vartheta_2) \leq \frac{3\beta^2}{2\delta}\|u\|^2\,\|v\|^2+\frac{\delta}{6}\|\vartheta_2\|^2-\beta\|Iu\|\,\|Iv\|\,\|\vartheta_2\|.$$

$$B(v,v,\vartheta_2)-B(Iv,Iv,\vartheta_2) \leq \frac{3\beta^2}{2\delta}\|\eta_2\|^4+\frac{\delta}{6}\|\vartheta_2\|^2.$$

Inserting these inequalities into (4.6) with using Cauchy-Schwartz inequality and Young's inequality to $\alpha(P_{L_H^\perp}\nabla\eta_1, P_{L_H^\perp}\nabla\vartheta_1)$ and $\alpha(P_{L_H^\perp}\nabla\eta_2, P_{L_H^\perp}\nabla\vartheta_2)$ we have,

$$\delta\|\vartheta_1\|^2+\alpha\|P_{L_H^\perp}\nabla\vartheta_1\|^2+\beta\|\vartheta_1\|^2\|\vartheta_1\|+\beta(\|pv_h\|\,\|pu_h\|-\|Iv\|\,\|Iu\|)\|\vartheta_1\|$$
$$\leq \frac{3\,C^2h^{-2}\epsilon^2}{2\delta}\|\nabla\eta_1\|^2+\frac{\delta}{6}\|\vartheta_1\|^2+\frac{\alpha}{4}\|P_{L_H^\perp}\nabla\eta_1\|^2+\alpha\|P_{L_H^\perp}\nabla\vartheta_1\|^2+\frac{3\beta^2}{2\delta}\|\eta_1\|^4+\frac{\delta}{6}\|\vartheta_1\|^2$$
$$+\frac{3\beta^2}{2\delta}\|v\|^2\,\|u\|^2+\frac{\delta}{6}\|\vartheta_1\|^2-\beta\|Iv\|\,\|Iu\|\,\|\vartheta_1\|,$$

$$\delta\|\vartheta_2\|^2+\alpha\|P_{L_H^\perp}\nabla\vartheta_2\|^2+\beta\|\vartheta_2\|^2\|\vartheta_2\|+\beta(\|pv_h\|\,\|pu_h\|-\|Iu\|\,\|Iu\|)\|\vartheta_2\|$$
$$\leq \frac{3\,C^2h^{-2}\epsilon^2}{2\delta}\|\nabla\eta_2\|^2+\frac{\delta}{6}\|\vartheta_2\|^2+\frac{\alpha}{4}\|P_{L_H^\perp}\nabla\eta_2\|^2+\alpha\|P_{L_H^\perp}\nabla\vartheta_2\|^2+\frac{3\beta^2}{2\delta}\|u\|^2\,\|v\|^2+\frac{\delta}{6}\|\vartheta_2\|^2$$
$$-\beta\|Iu\|\,\|Iv\|\,\|\vartheta_2\|+\frac{3\beta^2}{2\delta}\|\eta_2\|^4+\frac{\delta}{6}\|\vartheta_2\|^2,$$

Multiplying by two and rearranging gives,

$\delta\,\|\vartheta_1\|^2+2\beta\|\vartheta_1\|^2\|\vartheta_1\|+2\beta\|pv_h\|\ \|pu_h\|\,\|\vartheta_1\|\le\frac{3\,C^2h^{-2}\epsilon^2}{\delta}\|\nabla\eta_1\|^2+$
$\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla\eta_1\|^2+\frac{3\beta^2}{\delta}\|\eta_1\|^4\ +\frac{3\beta^2}{\delta}\|v\|^2\ \|u\|^2,$

$\delta\,\|\vartheta_2\|^2+2\beta\|\vartheta_2\|^2\|\vartheta_2\|+2\beta\|pv_h\|\ \|pu_h\|\,\|\vartheta_2\|\le\frac{3\,C^2h^{-2}\epsilon^2}{\delta}\|\nabla\eta_2\|^2+$
$\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla\eta_2\|^2+\frac{3\beta^2}{\delta}\|u\|^2\ \|v\|^2+\frac{3\beta^2}{\delta}\|\eta_2\|^4,$

Since $\beta\|\vartheta_1\|^2$, $\|\vartheta_1\|$, $\beta\|pv_h\|\,\|pu_h\|$, $\beta\|\vartheta_2\|^2$, $\|\vartheta_2\|$ and $\beta\|pv_h\|\,\|pu_h\|$ are nonnegative, we have,

$\|\vartheta_1\|^2\le C_1\{h^{-2}\epsilon^2\|\nabla\eta_1\|^2+\|P_{L_H^{\perp}}\nabla\eta_1\|^2+\|\eta_1\|^4+\|v\|^2\ \|u\|^2\},$

$\|\vartheta_2\|^2\le C_2\{h^{-2}\epsilon^2\|\nabla\eta_2\|^2+\|P_{L_H^{\perp}}\nabla\eta_2\|^2+\|\eta_2\|^4+\|u\|^2\ \|v\|^2\},$

Inside the brackets, from(2.5) we have,

$C_1\{h^{-2}\epsilon^2\|\nabla\eta_1\|^2+\|P_{L_H^{\perp}}\nabla\eta_1\|^2+\|\eta_1\|^4\}\le C_1\{(h^{-2}\epsilon^2+1)h^{2(r-1)}\|u\|_r^2+h^{4r}\|u\|_r^4\},$ (4.7.a)

$C_2\{h^{-2}\epsilon^2\|\nabla\eta_2\|^2+\|P_{L_H^{\perp}}\nabla\eta_2\|^2+\|\eta_2\|^4\}\le C_2\{(h^{-2}\epsilon^2+1)h^{2(r-1)}\|v\|_r^2+h^{4r}\|v\|_r^4\}$, (4.7.b)

So, $\|\vartheta_1\|^2_{L^\infty(L^2)}\le C_1\{(h^{4r}\|u\|^4_{L^\infty(H^r)}+(\epsilon^2h^{2(r-2)}+h^{2(r-1)})\|u\|^2_{L^\infty(H^r)}+\|v\|^2_{L^\infty(H^r)}\|u\|^2_{L^\infty(H^r)}\},$

$\|\vartheta_2\|^2_{L^\infty(L^2)}\le C_2\{(h^{4r}\|v\|_r^4+(\epsilon^2h^{2(r-2)}+h^{2(r-1)})\|v\|^2_{L^\infty(H^r)}+\|u\|^2_{L^\infty(H^r)}\|v\|^2_{L^\infty(H^r)}\},$

which implies

$\|\vartheta_1\|_{L^\infty(L^2)}\le C_1\{(h^{2r}\|u\|^2_{L^\infty(H^r)}+(\epsilon h^{r-2}+h^{r-1})\|u\|_{L^\infty(H^r)}+\|v\|_{L^\infty(H^r)}\|u\|_{L^\infty(H^r)}\}\ \le$
$C_3(h^{2r}+\epsilon h^{r-2}+h^{r-1}),$

$\|\vartheta_2\|_{L^\infty(L^2)}\le C_2\{(h^{2r}\|v\|^2_{L^\infty(H^r)}+(\epsilon h^{r-2}+\alpha^{\frac{-1}{2}}h^{r-1})\|v\|_{L^\infty(H^r)}+\|u\|_{L^\infty(H^r)}\|v\|_{L^\infty(H^r)}\}\ \le$
$C_4(h^{2r}+\epsilon h^{r-2}+h^{r-1}).$

Combining these bounds with (4.4) completes the proof.

Theorem 4.1: Let u, v, u_h and v_h be the solutions of (3.2) and (4.2) respectively, then there exists constants C_3, C_4 independent of independent of ϵ, α, h and H such that,

$$\|u - u_h\|_{L^\infty(L^2)} \le C_3\left(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1} + \sqrt{\alpha}\right),$$

$$\|v - v_h\|_{L^\infty(L^2)} \le C_4\left(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1} + \sqrt{\alpha}\right).$$

Proof: We write the errors in terms of equilibrium projection pu_h and pv_h

$u - u_h = (u - pu_h) - (u_h - pu_h) = \rho_1 - \theta_1$, $v - v_h = (v - pv_h) - (v_h - pv_h) = \rho_2 - \theta_2$, then ,

$\|u - u_h\|_{L^\infty(L^2)} \le \|\rho_1\|_{L^\infty(L^2)} + \|\theta_1\|_{L^\infty(L^2)}$, $\|v - v_h\|_{L^\infty(L^2)} \le \|\rho_2\|_{L^\infty(L^2)} + \|\theta_2\|_{L^\infty(L^2)}$.

By lemma (4.4) we have bounds on both $\|\rho_1\|_{L^\infty(L^2)}$ and $\|\rho_2\|_{L^\infty(L^2)}$. To estimate θ_1^n and θ_2^n, subtracting equation(4.2) from (3.2) we have,

$(\rho_{1,t} - \theta_{1,t}, \varphi_h) + a(\rho_1 - \theta_1, \varphi_h) - \alpha(P_{L_H^\perp} \nabla u_h, P_{L_H^\perp} \nabla \varphi_h) + (uu_x - u_h u_{h,x}, \varphi_h \quad (vu_y - v_h u_{h,y}, \varphi_h) = 0,$
(4.8.a)

$(\rho_{2,t} - \theta_{2,t}, \varphi_h) + a(\rho_2 - \theta_2, \varphi_h) - \alpha(P_{L_H^\perp} \nabla v_h, P_{L_H^\perp} \nabla \varphi_h) + (uv_x - u_h v_{h,x}, \varphi_h) + (v v_y - v_h v_{h,y}, \varphi_h) = 0,$
(4.8.b)

The third term on the left hand side of (4.8.a) and (4.8.b) can be rewritten using $\nabla u_h = \nabla\theta_1 - \nabla\rho_1 + \nabla u$, $\quad \nabla v_h = \nabla\theta_2 - \nabla\rho_2 + \nabla v$

also $uu_x - u_h u_{h,x} = \left(uu_x - pu_h pu_{h,x}\right) - \left(u_h u_{h,x} - pu_h\, pu_{h,x}\right)$,

$vu_y - v_h u_{h,y} = \left(vu_y - pv_h\, pu_{h,y}\right) - \left(v_h u_{h,y} - pv_h\, pu_{h,y}\right)$,

$uv_x - u_h v_{h,x} = \left(uv_x - pu_h\, pv_{x,h}\right) - \left(u_h v_{h,x} - pu_h\, pv_{x,h}\right)$ and
$vv_y - v_h v_{h,y} = \left(vv_y - pv_h\, pv_{h,y}\right) - \left(v_h v_{h,y} - pv_h\, pv_{h,y}\right)$, thus,

$(\theta_{1,t},\varphi_h)+a(\theta_1,\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla\theta_1,P_{L_H}^{\perp}\nabla\varphi_h)+(u_hv_{h,x}-pu_hpv_{h,x},\varphi_h)+(v_hv_{h,y}-pv_h\,pv_{h,y},\varphi_h)$
$=(\rho_{1,t},\varphi_h)\ +a(\rho_1,\varphi_h)+\ \alpha(P_{L_H}^{\perp}\nabla\rho_1,P_{L_H}^{\perp}\nabla\varphi_h)+(uv_x-pu_hpv_{h,x},\varphi_h)\ +(vv_y-pv_hpv_{h,y},\varphi_h)$
$-\alpha(P_{L_H}^{\perp}\nabla v,P_{L_H}^{\perp}\nabla\varphi_h),$ (4.9.a)

$(\theta_{2,t},\varphi_h)+a(\theta_2,\varphi_h)+\alpha(P_{L_H}^{\perp}\nabla\theta_2,P_{L_H}^{\perp}\nabla\varphi_h)+(u_hv_{h,x}-pu_hpv_{h,x},\varphi_h)\ +(v_hv_{h,y}-pv_h\,pv_{h,y},\varphi_h$
$)=(\rho_{2,t},\varphi_h)\ +a(\rho_2,\varphi_h)+\ \alpha(P_{L_H}^{\perp}\nabla\rho_2,P_{L_H}^{\perp}\nabla\varphi_h)+(uv_x-pu_hpv_{h,x},\varphi_h)\ +(vv_y-pv_hpv_{h,y},\varphi_h)$
$-\alpha(P_{L_H}^{\perp}\nabla v,P_{L_H}^{\perp}\nabla\varphi_h),$ (4.9.b)

From (4.3), $A(\rho_1,\varphi_h)=0$, $A(\rho_2,\varphi_h)=0$ and by choosing $\varphi_h=\theta_1$, and $\varphi_h=\theta_2$ in (4.9.a) and (4.9.b) respectively we have,

$(\theta_{1,t},\theta_1)+a(\theta_1,\theta_1)+\alpha(P_{L_H}^{\perp}\nabla\theta_1,P_{L_H}^{\perp}\nabla\theta_1)+(u_hu_{h,x}-pu_hpu_{h,x},\theta_1)+\ (v_hu_{h,y}-pv_hpu_{h,y}\ ,\theta_1)\ =$
$(\rho_{1,t}\ ,\theta_1)-\alpha(P_{L_H}^{\perp}\nabla u,P_{L_H}^{\perp}\nabla\theta_1),$

$(\theta_{2,t},\theta_2)+a(\theta_2,\theta_2)+\alpha(P_{L_H}^{\perp}\nabla\theta_2,P_{L_H}^{\perp}\nabla\theta_2)+(u_hv_{h,x}-pu_hpv_{h,x}\ ,\theta_2)\ +(v_hv_{h,y}-pv_hpv_{h,y},\theta_2)\ =$
$(\rho_{2,t}\ ,\theta_2)-\alpha(P_{L_H}^{\perp}\nabla v,P_{L_H}^{\perp}\nabla\theta_2),$

Since, $(\theta_{1,t},\theta_1)=\frac{1}{2}\frac{d}{dt}\|\theta_1\|^2$, $(\theta_{2,t},\theta_2)\ \frac{1}{2}\frac{d}{dt}\|\theta_2\|^2$, also from(2.3) , $a(\theta_1,\theta_1)\geq\delta\|\vartheta_1\|^2$, $a(\theta_2,\theta_2)\geq\delta\|\vartheta_2\|^2$, from(2.4),$|(u_hu_{h,x}-pu_hpu_{h,x},\theta_1)|\leq\beta(\|u_h\|^2+\|pu_h\|^2)\|\vartheta_1\|$, similarly, $|(v_hu_{h,y}-pv_hpu_{h,y},\theta_1)|\leq\beta(\|v_h\|\,\|u_h\|+\|pv_h\|\,\|pu_h\|)\|\vartheta_1\|$,

$|(u_hv_{h,x}-pu_h\,pv_{h,x},\theta_2)|\leq\beta(\|u_h\|\ \|v_h\|+\|pu_h\|\ \|pv_h\|)\|\vartheta_2\|$

and $|(v_hv_{h,y}-pv_h\,pv_{h,y},\theta_2)|\leq\beta(\|v_h\|^2+\|pv_h\|^2)\|\vartheta_2\|$.

Using Cauchy-Schwartz inequality and Young's inequality for the right hand sides we have ,

$\frac{1}{2}\frac{d}{dt}\|\theta_1\|^2+\delta\|\vartheta_1\|^2+\alpha\|P_{L_H}^{\perp}\nabla\vartheta_1\|^2+\beta(\|u_h\|^2+\|pu_h\|^2)\|\vartheta_1\|\ +\beta(\|v_h\|\ \|u_h\|+$
$\|pv_h\|\ \|pu_h\|)\|\vartheta_1\|\ \leq\frac{1}{2\delta}\|\rho_{1,t}\|^2+\frac{\delta}{2}\|\theta_1\|^2+\frac{\alpha}{4}\|P_{L_H}^{\perp}\nabla u\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_1\|^2,$

$\frac{1}{2}\frac{d}{dt}\|\theta_2\|^2+\delta\|\theta_2\|^2+\alpha\|P_{L_H^{\perp}}\nabla\theta_2\|^2+\beta(\|u_h\|\ \|v_h\|+\|pu_h\|\ \ \|pv_h\|)\ \ \|\vartheta_2\|+\beta(\|v_h\|^2+\|pv_h\|^2)\|\vartheta_2\| \quad \leq\frac{1}{2\delta}\|\rho_{2,t}\|^2+\frac{\delta}{2}\|\theta_2\|^2+\frac{\alpha}{4}\|P_{L_H^{\perp}}\nabla v\|^2+\alpha\|P_{L_H^{\perp}}\nabla\theta_2\|^2,$

since, $\|u_h\|^2, \|pu_h\|^2, \|v_h\|\ \|u_h\|, \|pv_h\|\ \|pu_h\|, \|v_h\|^2, \|pv_h\|^2, \|\vartheta_1\|$ and $\|\vartheta_2\|$ are nonnegative terms , by multiplying by two and rearranging we get,

$\frac{d}{dt}\|\theta_1\|^2+\delta\|\theta_1\|^2 \ \leq\frac{1}{\delta}\|\rho_{1,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla u\|^2, \frac{d}{dt}\|\theta_2\|^2+\delta\|\theta_2\|^2 \ \leq\frac{1}{\delta}\|\rho_{2,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla v\|^2,$

using the integrating factor $e^{\delta t}$, integrating from $t = 0$ to $t = T$ and rearranging give

$$e^{\delta T}\|\theta_1(T)\|^2 \leq \|\theta_1(0)\|^2+\int_0^T e^{\delta t}\left[\frac{1}{\delta}\|\rho_{1,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla u\|^2\right]dt,$$

$$e^{\delta T}\|\theta_2(T)\|^2 \leq \|\theta_2(0)\|^2+\int_0^T e^{\delta t}\left[\frac{1}{\delta}\|\rho_{2,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla v\|^2\right]dt,$$

multiplying by $e^{-\delta T}$gives

$$\|\theta_1(T)\|^2 \leq \ e^{-\delta T}\|\theta_1(0)\|^2+\int_0^T e^{\delta(t-T)}\left[\frac{1}{\delta}\|\rho_{1,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla u\|^2\right]dt, \qquad (4.10.a)$$

$$\|\theta_2(T)\|^2 \leq \ e^{-\delta T}\|\theta_2(0)\|^2+\int_0^T e^{\delta(t-T)}\left[\frac{1}{\delta}\|\rho_{2,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla v\|^2\right]dt, \qquad (4.10.b)$$

note that, there exist $0\leq t^*\leq T$ such that, $\|\theta_1(t^*)\| = \|\theta_1\|_{L^\infty(L^2)}$ and $\|\theta_2(t^*)\| = \|\theta_2\|_{L^\infty(L^2)}$,

such that , $\|\theta_1\|^2_{L^\infty(L^2)}\leq e^{-\delta T}\|\theta_1(0)\|^2+\int_0^T e^{\delta(t-T)}\left[\frac{1}{\delta}\|\rho_{1,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla u\|^2\right]dt,$

$$\|\theta_2\|^2_{L^\infty(L^2)}\leq e^{-\delta T}\|\theta_2(0)\|^2+\int_0^T e^{\delta(t-T)}\left[\frac{1}{\delta}\|\rho_{1,t}\|^2+\frac{\alpha}{2}\|P_{L_H^{\perp}}\nabla u\|^2\right]dt,$$

as the above implies that each of $\|\theta_1\|^2_{L^\infty(L^2)}$ and $\|\theta_2\|^2_{L^\infty(L^2)}$ are bounded by the right hand side, we can write,

$$\|\theta_1\|_{L^\infty(L^2)}\leq C\{\|\theta_1(0)\|+\left(\int_0^T\|\rho_{1,t}\|^2\,dt\right)^{\frac{1}{2}}+\sqrt{\alpha}\left(\int_0^T\|P_{L_H^{\perp}}\nabla u\|^2\,dt\right)^{\frac{1}{2}}\} \leq C\{\|\theta_1(0)\|+\|\rho_{1,t}\|_{L^2(L^2)}+\sqrt{\alpha}\|P_{L_H^{\perp}}\nabla u\|_{L^2(L^2)}\},$$

$$\|\theta_2\|_{L^\infty(L^2)} \le C\{\|\theta_2(0)\| + (\int_0^T \|\rho_{2,t}\|^2 dt)^{\frac{1}{2}} + \sqrt{\alpha}(\int_0^T \|P_{L_H}^{\perp} \nabla v\|^2 dt)^{\frac{1}{2}}\} \le C\{\|\theta_2(0)\| + \|\rho_{2,t}\|_{L^2(L^2)} + \sqrt{\alpha}\|P_{L_H}^{\perp} \nabla v\|_{L^2(L^2)}\}.$$

The first terms on right hand sides give,

$$\|\theta_1(0)\| \le \|u_h^0 - pu_h^0\| \le \|u_h^0 - u^0\| + \|u^0 - pu_h^0\|$$

$$\le \|u_h^0 - u^0\| + Ch^r\|u^0\|_r, \text{ from (A2) we have } \|\theta_1(0)\| \le C_1 h^r, \qquad (4.11.a)$$

$$\|\theta_2(0)\| \le \|v_h^0 - pv^0\| \le \|v_h^0 - v^0\| + \|v^0 - pv_h^0\|$$

$$\le \|v_h^0 - v^0\| + Ch^r\|v^0\|_r \le C_2 h^r. \qquad (4.11.b)$$

Form Lemma 4.4 the second terms imply ,

$$\|\rho_{1,t}\|_{L^2(L^2)} = \|u_t - pu_{h,t}\|_{L^2(L^2)} \le \|\eta_{1,t}\|_{L^2(L^2)} + \|\vartheta_{1,t}\|_{L^2(L^2)}$$

$$\le C\{(h^{2r}\|u_t\|_{L^2(H^r)}^2 + (\epsilon h^{r-2} + h^{r-1})\|u_t\|_{L^2(H^r)} + \|v_t\|_{L^2(H^r)}\|u_t\|_{L^2(H^r)}\},$$

$$\le C_1(h^{2r} + \epsilon h^{r-2} + h^{r-1}), \qquad (4.12.a)$$

$$\|\rho_{2,t}\|_{L^2(L^2)} = \|v_t - pv_{h,t}\|_{L^2(L^2)} \le \|\eta_{2,t}\|_{L^2(L^2)} + \|\vartheta_{2,t}\|_{L^2(L^2)}$$

$$\le C\{(h^{2r}\|v_t\|_{L^2(H^r)}^2 + (\epsilon h^{r-2} + h^{r-1})\|v_t\|_{L^2(H^r)} + \|u_t\|_{L^2(H^r)}\|v_t\|_{L^2(H^r)}\},$$

$$\le C_2(h^{2r} + \epsilon h^{r-2} + h^{r-1}). \qquad (4.12.b)$$ then ,

$$\|\theta_1\|_{L^\infty(L^2)} \le C_1[h^r + (h^{2r} + \epsilon h^{r-2} + h^{r-1}) + \sqrt{\alpha}\|P_{L_H}^{\perp} \nabla u\|_{L^2(L^2)}],$$ from (A2) we have,

$$\|\theta_1\|_{L^\infty(L^2)} \le C_3(h^r + h^{2r} + \epsilon h^{r-2} + h^{r-1} + \sqrt{\alpha}), \qquad (4.13.a)$$

$$\|\theta_2\|_{L^\infty(L^2)} \le C_2[h^r + (h^{2r} + (\sqrt{\epsilon}\, h^{r-2} + h^{r-1}) + \sqrt{\alpha}\|P_{L_H}^{\perp} \nabla u\|_{L^2(L^2)}]$$

$$\le C_4(h^r + (h^{2r} + \sqrt{\epsilon}\, h^{r-2} + h^{r-1} + \sqrt{\alpha}). \qquad (4.13.b)$$

Combining the bound (4.13) with those of lemma (4.4) the proof complete.

***5.* The Finite Element Approximation**

To approximate the artificial viscosity terms in the equation (4.2), note that,

$\nabla\varphi_h = P_{L_H}\nabla\varphi_h + P_{L_H^{\perp}}\nabla\varphi_h$, $P_{L_H^{\perp}}\nabla\varphi_h = \nabla\varphi_h - P_{L_H}\nabla\varphi_h$, then

$$\alpha\left(P_{L_H^{\perp}}\nabla u_h , P_{L_H^{\perp}}\nabla\varphi_h\right) = \alpha(P_{L_H^{\perp}}\nabla u_h, \nabla\varphi_h) - \alpha(P_{L_H^{\perp}}\nabla u_h , P_{L_H}\nabla\varphi_h)$$

From the definition of $L_H^{\perp}$ the second term equal to zero, this implies

$\alpha(P_{L_H^{\perp}}\nabla u_h, P_{L_H^{\perp}}\nabla\varphi_h) \equiv \alpha(P_{L_H^{\perp}}\nabla u_h, \nabla\varphi_h)$, similarly, $\alpha(P_{L_H^{\perp}}\nabla v_h, P_{L_H^{\perp}}\nabla\varphi_h) \equiv \alpha(P_{L_H^{\perp}}\nabla v_h, \nabla\varphi_h)$.

The main point comes in finding an appropriate interpretation of the $\alpha(P_{L_H^{\perp}}\nabla u_h, \nabla\varphi_h)$ and $\alpha(P_{L_H^{\perp}}\nabla v_h, \nabla\varphi_h)$ terms. Since , $\nabla u_h = P_{L_H}\nabla u_h + P_{L_H^{\perp}}\nabla u_h$, $\nabla v_h = P_{L_H}\nabla v_h + P_{L_H^{\perp}}\nabla v_h$,

We rewrite, $\alpha\left(P_{L_H^{\perp}}\nabla u_h, \nabla\varphi_h\right) = \alpha(\nabla u_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla u_h, \nabla\varphi_h\right)$,

$$\alpha\left(P_{L_H^{\perp}}\nabla v_h, \nabla\varphi_h\right) = \alpha(\nabla v_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla v_h, \nabla\varphi_h\right).$$

As noted in section four, L_H is chosen such that $P_{L_H^{\perp}}$ is a projection onto fine scales and P_{L_H} is a projection onto the large scales, we can think of the large scale as representing average values, this implies, $\alpha(\nabla u_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla u_h, \nabla\varphi_h\right) \approx \alpha(\nabla u_h, \nabla\varphi_h) - \alpha(\nabla\bar{u}_h, \nabla\varphi_h)$,

$$\alpha(\nabla v_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla v_h, \nabla\varphi_h\right) \approx \alpha(\nabla v_h, \nabla\varphi_h) - \alpha(\nabla\bar{v}_h, \nabla\varphi_h).$$

Where $\bar{u}_h$ and $\bar{v}_h$ is an average over itself and its five nearest discrete neighbors.

5.1 Test problem

In this subsection ,we present the test problem to illustrate G.P.A.D. for the time - dependent Burgers' equation (4.2). The exact solution of Burgers' equation(3.1) can be generated by using the Hopf–Cole transformation (*see* [9]) which are :

$$u(x, y, t) = \frac{3}{4} - \frac{1}{4\left[1+e^{\frac{(-4x+4y-t)}{32\epsilon}}\right]}, \quad v(x, y, t) = \frac{3}{4} + \frac{1}{4\left[1+e^{\frac{(-4x+4y-t)}{32\epsilon}}\right]}.$$

In this problem ϵ can take on various values and $f = g = 0$. The domain Ω where the problem is to be solved is the unit square domain $\bar{\Omega}$ = [0,1] × [0,1]. We are discretized it using a uniform triangular mesh with mesh width parameter $h=\frac{1}{N-1}$ where we take N =18 .

5.2 Numerical Results

This subsection consists of two case was discussed as follows:

Case 1: In this case the problem was run with ϵ=1.14 at $t = 0.5$, we note that $\epsilon > h$, there is no need to run this problem with P.A.D.(i.e $\alpha = 0$), the numerical solution of the standard Galerkin(G.) finite element method are convergent to the exact solution{*see* Figure (5.2.1)}.

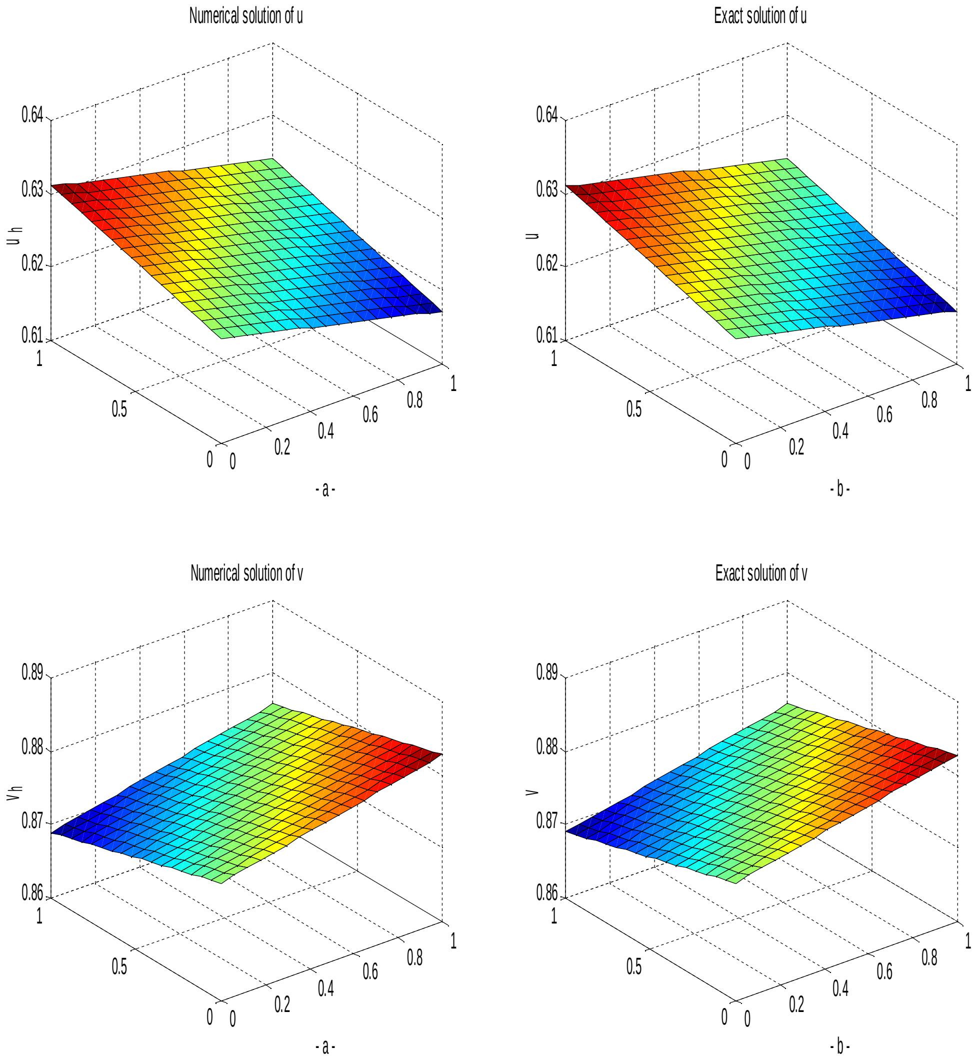

Figure 5.2.1: a- *Numerical solution of G. method of u and v,* b-*Exact solution of u and v,*

at N=18, t=.5 and ϵ=1.14 .

Case 2: In this case we take $\epsilon=\frac{1}{120}$ and $\epsilon=\frac{1}{240}$ respectively at $t = 0.5$, we note that $\epsilon < h$, thus we expect to see instability for standard G. finite element method, in Figure{(5.2.2)(a) and (5.2.3)(a)}the problem run without P.A.D.(i.e. $\alpha = 0$), we see that the standard G. finite element method produce an oscillating solution which is not close to the exact solution especially when ϵ decreasing with respect to h. In Figure{(5.2.2)(b) and (5.2.3)(b) the problem run with G.P.A.D. finite element method. where, $\alpha = 0.25 * h$, where the numerical solution became more convergent to the exact solution.

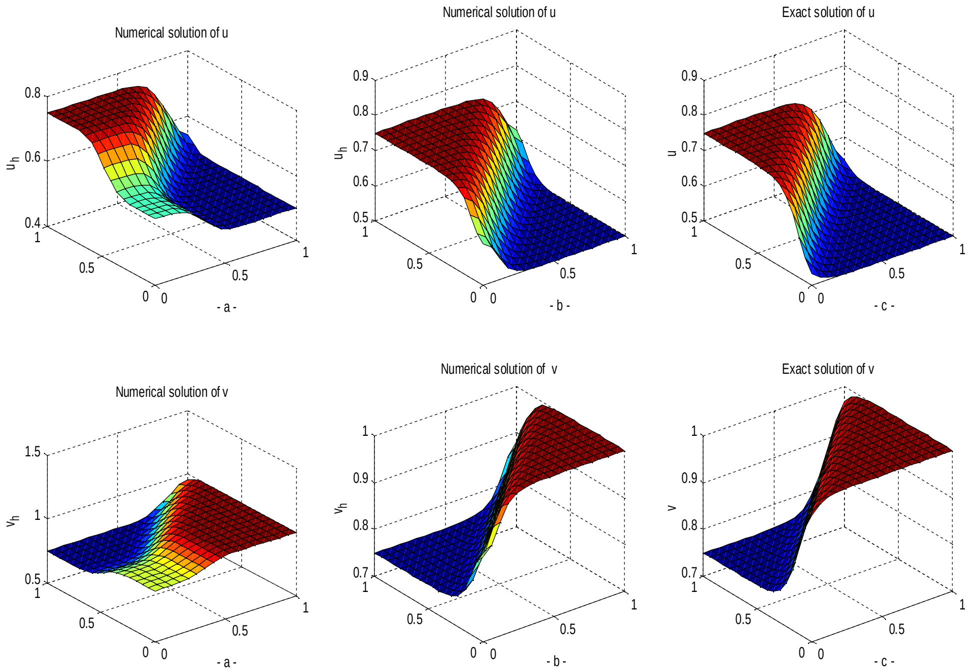

Figure 5.2.2 : a- *Numerical solution of G. method without P.A.D. of* u *and* v, b- *Numerical solution of G.P.A.D. of* u *and* v, c- *Exact solution of* u *and* v, *at N=18,t=.5 and* $\epsilon=\frac{1}{120}$.

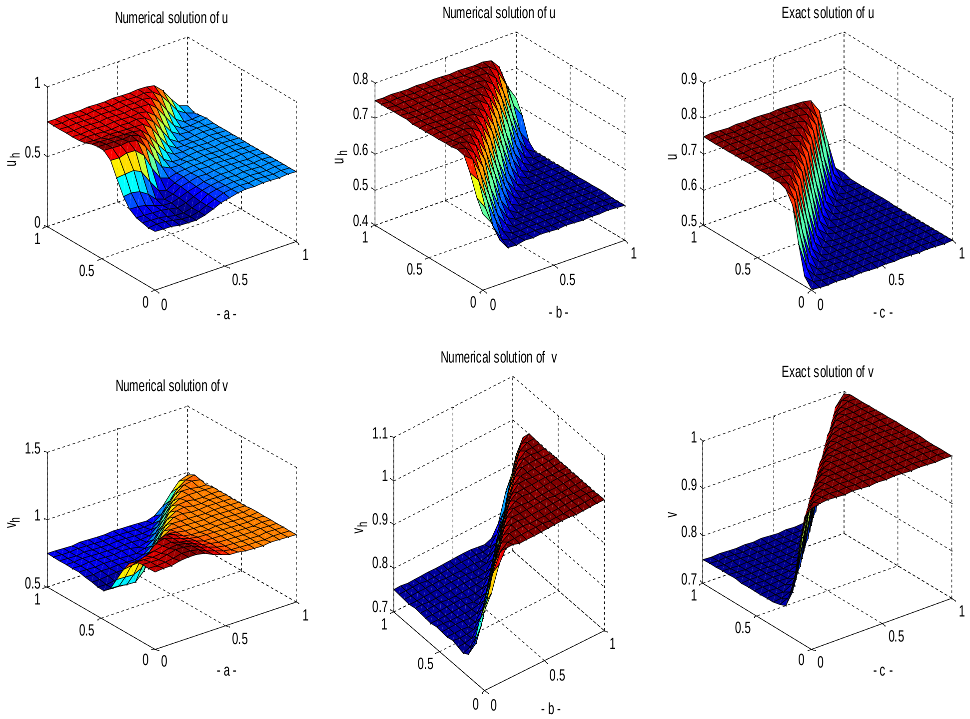

Figure 5.2.3 : a- *Numerical solution of G. method without P.A.D. of* u *and* v, b- *Numerical solution of G.P.A.D. of* u *and* v, c- *Exact solution of* u *and* v, *at N=18,* t*=.5 and* $\epsilon=\frac{1}{240}$.

6 - CONCLUSIONS

From the theoretical analysis and the numerical results ,we can conclude the following :

1-The continuity and V-elliptic of $A(u,\varphi)$ and $A(v,\varphi)$ is satisfied.

2-The G.P.A.D. finite element method satisfy the stability .

3-Theoretical analysis shows that G.P.A.D. finite element method are convergent with $O(h^{2r})$.

4-The G.P.A.D. method removed all oscillations occur when we use the standard Galerkin in the convection-dominated case, and the numerical solutions obtained from this method are consistent with the exact solution.

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