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ERROR ESTIMATE FOR SPACE-TIME DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD OF CONVECTION-DIFFUSION PROBLEM

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Abstract: This paper presents the theory of the space-time discontinuous Galerkin finite element (DGFE) method for linear convection – diffusion problem. DGFE method is applied separately in space and time using, in general, different space grids on different time levels. We prove the properties of the bilinear form $A(u, v)$, (v-elliptic and continuity), stability and prove the approximate solution is converges with error of order $O(h^{r+1} + \tau^{s+1})$.

Keywords: linear convection – diffusion equation, discontinuous Galerkin method, convergent, stability.

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INTRODUCTION

A number of complex problems from science and technology (aerospace engineering, turbo machinery, oil recovery, meteorology, environmental protection etc.) require to apply new efficient, robust, reliable and highly accurate numerical methods. It is necessary to develop techniques that allow realizing numerical approximations of strongly nonlinear singularly perturbed systems in domains with complex geometry whose solution contain internal or boundary layers. An excellent candidate to overcome these difficulties is the DGFE method, which becomes more and more popular in the solution of a number of problems. The DGFE method uses piecewise polynomial approximations of the sought solution on a finite elements mesh without any requirement on continuity between neighboring elements and can be considered a generalization of the finite volume and finite element methods . It allows to construct higher order schemes in a natural way and is suitable for approximation of discontinuous solutions of conservation laws or solutions of singularly perturbed convection-diffusion problems having steep gradients. This method exploits' advantages of the finite element method and finite volume schemes with an approximate Riemann solver and can be applied on unstructured grids which are generated for most complex geometries. The original DGFE method was introduced in [8] for the solution of a neutron transport linear equation and analyzed theoretically in [7] and later in [6]. Almost simultaneously the DGFE techniques were developed for the numerical solution of elliptic problems [14] and space semi discretization of parabolic problems [5], [1], using the interior penalty Galerkin methods. In the discretization of non stationary problems, one often uses the space semi discretization, also called the method of lines. The DGFE discretization with respect to space leads to a large system of ordinary differential equations which can be solved numerically by a suitable ODE solver (See, e.g., [9], [10], [3], [4], [2], [11], [12], [13]).

In the present paper we are concerned with the space-time discontinuous Galekin discretization applied to the convection-diffusion problem. The time interval is split into subintervals and on each time level a different space mesh may be used in general. Moreover, the triangulations used for the space discretization may be nonconforming with hanging nodes.

This paper is organized as follows. In section 2 we present the convection-diffusion problem. Some definitions in section 3. The discretization is shown in section 4. In section 5 derive the Weak form. In section 6 we proved the properties of the bilinear form and stability. The error estimate are presented in section 7. The conclusion are shown in section 8.

2. The Convection-Diffusion problem.

Let $\Omega \subset R^2$ be a bounded polyhedral domain and T > 0. We consider he convection-diffusion problem[18]: Find $u \in Q_T = \Omega \times (0,T) \to R$such that

$$u_t - a\Delta u + b\nabla u = f \qquad in \ Q_T \tag{2.1}$$

$$u = u_D \qquad on \quad \partial\Omega_D \times (0,T) \tag{2.2}$$

$$a\frac{\partial u}{\partial n} = u_N \qquad on \quad \partial\Omega_N \times (0,T) \tag{2.3}$$

$$u(x,0) = u^0(x), \qquad x \in \Omega\,. \tag{2.4}$$

We assume that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$

$$b \cdot n < 0 \quad on \quad \partial\Omega_D \tag{2.5}$$

$$b \cdot n \geq 0 \qquad on \quad \partial\Omega_N \text{ for all } t \in [0\,.T], \tag{2.6}$$

here n is the unit outer normal to the boundary $\partial\Omega$ of $\Omega, \partial\Omega_D$ is the inflow boundary and $\partial\Omega_N$ is the outflow boundary

3-Definitions:

It is beneficial to mention the definitions of the vector space that we used during this study. The vector space $L^2(\Omega)$ is the space of square-integrable functions on $\Omega \subset R^2$

$$L^2(\Omega) = \left\{v: \Omega \to R \ s.t. \ \int v^2 \, d\Omega \leq \infty\right\},$$

indeed $L^2(\Omega)$ is Hilbert space with respect to the following inner product

$$(u,v) = \int_\Omega u(x)\, v\,(x)\, dx \text{ and norm } \|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 d\,\Omega\right)^{\frac{1}{2}}$$

for $p = \infty$, $L^\infty(\Omega)$ denotes the space of all functions which are bounded for almost all $x \in \Omega$:

$$L^\infty(\Omega) = \{u : |u(x)| < \infty \ for \ almost \ all \ x \in \Omega\},$$

this space is equipped with the norm

$$\|v\|_{L^\infty(\Omega)} = \{ess \sup\{|v(x)|: x \in R\}.$$

We introduce the sobolev space

$$H^1(\Omega) = \left\{v \in L^2(\Omega): \frac{\partial v}{\partial x_i} \in L^2(\Omega), \qquad i = 1,2 \ldots \ldots d\right\}$$

and the corresponding norm,

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega (v^2 + (\nabla v)^2 d\Omega)\right)^{\frac{1}{2}}$$

also,

$H_0^1(\Omega) = \{v \in H^1(\Omega): v = 0 \;\; on \;\; \partial\Omega\}$,

with the same scalar product and norm as $H^1(\Omega)$. We introduce the norm for both continuous time $t \in [0,T]$ and space Ω by:

$$\|v\|_{L^\infty(H^r(\Omega))} = \max_{0\le t\le T}\|v\|_r \;\; \text{and} \;\; \|v_t\|_{L^2(L^2(\Omega))} = \left(\int_0^t \|v_t\|^2\right)^{\frac{1}{2}}$$

We also use the Buchner spaces. Let X be a Banach space with a norm $\|\,.\,\|_X$ and a semi norm $|\,.\,|_X$. Then we define:

$$C([0,T];\, X) = \left\{v: [0,T] \to X, continuous, \|v\|_{C([0,T];\, X)} = \sup_{t\in[0,T]} \|v\|_X < \infty\right\},$$

$$L^2(0,T;\, X) = \{v: (0,T) \to X, strongly\ measurable, \|v\|^2_{L^2(0,T;\, X)} = \int_0^T \|v\|_X^2 dt < \infty\},$$

$$H^r(0,T;\, X) = \{v \in L^2(0,T;\, X);\; \|v\|^2_{H^r(0,T;\, X)} = \int_0^T \sum_{\alpha=0}^{r} \left\|\frac{\partial^\alpha v}{\partial t^\alpha}\right\|_X^2 dt < \infty\},$$

moreover, we set

$$|v|_{C([0,T];\, X)} = \sup_{t\in[0,T]} |v|_X, \;\; |v|^2_{L^2(0,T;\, X)} = \left(\int_0^T \|v\|_X^2 dt\right)^{\frac{1}{2}},$$

$$|v|^2_{H^r(0,T;\,X)} = \left(\int_0^T \sum_{\alpha=0}^{r} \left| \frac{\partial^\alpha v}{\partial t^\alpha} \right|_X^2 dt \right)^{\frac{1}{2}}.$$

4. Discretization of the problem:

In the time interval $t \in [0, T]$ we shall construct a partition $0 = t_0 < t_1 < \cdots < t_M = T$, and denote $I_m = (t_{m-1}, t_m)$, $\tau_m = t_m - t_{m-1}$, $\tau = max_{m=1,\cdots,M}\, \tau_m$. For each I_m we consider a partition $\tau_{h,m}$ of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles with mutually disjoint interiors. The partitions $\tau_{h,m}$ are in general different for different m. By $\partial\tau_{h,m}$ we denote the system of all edges e of all elements $E \in \tau_{h,m}$. Further, we denote the set of all inner and boundary edges by:

$\partial\tau^I_{h,m} = \{e \in \partial\tau_{h,m},\ e \subset \Omega\}$,

$\partial\tau^B_{h,m} = \{e \in \partial\tau_{h,m},\ e \subset \partial\Omega\}$,

$\Gamma_D = \{e \in \partial\tau^B_{h,m},\ e \subset \partial\Omega_D\}$,

$\Gamma_N = \{e \in \partial\tau^B_{h,m},\ e \subset \partial\Omega_N\}$,

For $\varphi \in H^1(\Omega, \tau_{h,m})$, we introduce the following notation. Obviously $\tau_{h,m} = \partial\tau^I_{h,m} \cup \partial\tau^B_{h,m}$, $\partial\tau^B_{h,m} = \Gamma_D \cup \Gamma_N$ for each $e \in \partial\tau_{h,m}$. For a function φ defined in $\cup_{m=1}^{M} I_m$.

Put, $\varphi^+_m = \varphi(t_{m+}) = lim_{t\to t_{m+}}\, \varphi(t)$, $\varphi^-_m = lim_{t\to t_{m-}}\, \varphi(t)$, $[\varphi]_m = (\varphi^+_m - \varphi^-_m)$,

$$\{\varphi\}_e = \frac{1}{2}(\varphi^+ + \varphi^-) \qquad (4.1)$$

$$[\varphi]_e = (\varphi^+ - \varphi^-) \qquad (4.2)$$

further,

$$\partial E^i_- = \{x \in \partial E^i : b \cdot n < 0\} \qquad (4.3)$$

$$\partial E^i_+ = \{x \in \partial E^i : b \cdot n \geq 0\} \qquad (4.4)$$

where n denotes the unit outer normal to ∂E^i.

4.1 Assumptions:

a) $f \in C([0, T]; L^2(\Omega))$, $u^0, u, u_t \in L^2(\Omega)$,

b) u_D is the trace of some $u \in C\left([0,T]; H^1(\Omega)\right) \cap L^\infty(QT)$ on $\partial\Omega_D \times (0,T)$

c) $u_N \in C\left([0,T]; L^2(\partial\Omega_N)\right)$,

d) $|E|$ = the area of $E \in \tau_{h,m}$, and $\sigma = \frac{\sigma^0}{|e|^{\beta_0}}$, $\beta_0 \geq (d-1)^{-1}$.

e) Define h_E = the length of the longest side of the triangle $E \in \tau_{h,m}$ and put h_E= diameter of $E. h = max_{E\in\tau_{h,m}} h_E$.

5. The weak form of problem.

We multiply equation (2.1) by the test function $v \in V = H^1(\Omega)$ and integrating by part such that:

$$\int_{I_m}\int u_t\, v\, d\Omega\, dt + \int_{I_m}\left(\sum_{E\in\tau_{h,m}}\int a\nabla u\nabla v\, d\Omega - \sum_{e\in\partial\tau_{h,m}}\int a\nabla u\cdot nv\, ds + \sum_{E\in\tau_{h,m}}\int b\cdot\nabla uv\, d\Omega - \sum_{e\in\partial\tau_{h,m}}\int |b\cdot n|uv\, ds\right)dt$$

$$= \int_{I_m}\left(\sum_{E\in\tau_{h,m}}\int fv\, d\Omega\right) dt.$$

where n denotes the outward normal to each element edge. The fourth and sixth terms in the left-hand side contains the integrals over the element edges, Then we have,

$$\int_{I_m}\int u_t\, v\, d\Omega\, dt + \int_{I_m}\left(\sum_{E\in\tau_{h,m}}\int a\nabla u\nabla v\, d\Omega - \sum_{e\in\partial\tau_{h,m}}\int [a\nabla u\cdot nv]\, ds + \sum_{E\in\tau_{h,m}}\int b\cdot\nabla uv\, d\Omega - \sum_{e\in\partial\tau_{h,m}}\int [|b\cdot n|uv]\, ds\right)dt = \int_{I_m}\left(\sum_{E\in\tau_{h,m}}\int fv\, d\Omega\right)dt$$

since $[uv] = \{u\}[v] + \{v\}[u]$ we have,

$$\int_{I_m}\int u_t\, v\, d\Omega\, dt + \int_{I_m}\Bigg(\sum_{E\in\tau_{h,m}}\int a\nabla u\nabla v\, d\Omega$$
$$-\sum_{e\in\partial\tau_{h,m}}\int([a\nabla u\cdot n]\{v\} + [v]\{a\nabla u\cdot n\})\, ds + \sum_{E\in\tau_{h,m}}\int b\cdot\nabla u v\, d\Omega$$
$$-\sum_{e\in\partial\tau_{h,m}}\int|b\cdot n|([u]\{v\} + \{u\}[v])ds\Bigg)dt = \int_{I_m}\Bigg(\sum_{E\in\tau_{h,m}}\int f v\, d\Omega\Bigg)dt.$$

since u is continuous then $[u]$ and $[a\nabla u\cdot n] = 0$, we get,

$$\int_{I_m}\int u_t\, v\, d\Omega\, dt + \int_{I_m}\Bigg(\sum_{E\in\tau_{h,m}}\int a\nabla u\nabla v\, d\Omega - \sum_{e\in\partial\tau_{h,m}}\int[v]\{a\nabla u\cdot n\}\, ds$$
$$-\sum_{E\in\tau_{h,m}}\int b\cdot\nabla u v\, d\Omega - \sum_{e\in\partial\tau_{h,m}}\int|b\cdot n|\{u\}[v]\, ds\Bigg)dt = \int_{I_m}\Bigg(\sum_{E\in\tau_{h,m}}\int f v\, d\Omega\Bigg)dt.$$

We note that the left hand side of the above equation is still non-symmetric and non positivity with respect to argument u and v [15], to rectify these properties, we add the terms,

$\varepsilon\sum_{e\in\partial\tau_{h,m}}\int[u]\{a\nabla v\cdot n\}\, ds$ and $\sigma\sum_{e\in\partial\tau_{h,m}}\int[u][v]ds,$

we have,

$$\int_{I_m}(u_t, v)dt + ([u]_{m-1}, v^+_{m-1}) + \int_{I_m}\Bigg(\sum_{E\in\tau_{h,m}}a(\nabla u,\nabla v)$$
$$-\sum_{e\in\partial\tau_{h,m}}\int(\{a\nabla u\cdot n\}[v] - \varepsilon\{a\nabla v\cdot n\}[u])\, ds + \sum_{E\in\tau_{h,m}}(b\cdot\nabla u, v)$$
$$-\sum_{e\in\partial\tau_{h,m}}\int|b\cdot n|\{u\}[v]\, ds + \sigma\sum_{e\in\partial\tau_{h,m}}\int[u][v]ds\Bigg)dt = \int_{I_m}\Bigg(\sum_{E\in\tau_{h,m}}(f, v)\Bigg)dt.$$

$$\int_{I_m} (u_t, v)dt + ([u]_{m-1}, v_{m-1}^+) + \int_{I_m} \Bigg(\sum_{E\in\tau_{h,m}} a(\nabla u, \nabla v)$$
$$- \sum_{e\in\partial\tau_{h,m}} \int (\{a\nabla u \cdot n\}[v] - \varepsilon\{a\nabla v \cdot n\}[u])\, ds + \sum_{E\in\tau_{h,m}} (b \cdot \nabla u, v)$$
$$- \sum_{e\in\partial\tau_{h,m}} \int |b \cdot n|\{u\}[v]\, ds + \sigma \sum_{e\in\partial\tau_{h,m}} \int [u][v] ds \Bigg) dt$$
$$= \int_{I_m} \Bigg(\sum_{E\in\tau_{h,m}} (f, v) + \sum_{e\in\Gamma_N} \int u_N\, v ds - \sum_{e\in\Gamma_D} \int \varepsilon a\nabla v \cdot n u_D\, ds$$
$$+ \sum_{e\in\Gamma_D} \int |b \cdot n| u_D v\, ds - \sigma \sum_{e\in\Gamma_D} \int u_D v ds \Bigg) dt. \qquad (5.1)$$

Then the weak form is: find $u \in V$ such that:

$$\int_{I_m} (u_t, v)\, dt + ([u]_{m-1}, v_{m-1}^+) + \int_{I_m} A_m(u, v)\, dt = \int_{I_m} B_m(v)\, dt \qquad (5.2)$$

where,

$$A_m(u, v) = \sum_{E\in\tau_{h,m}} a(\nabla u, \nabla v) - \sum_{e\in\partial\tau_{h,m}} \int (\{a\nabla u \cdot n\}[v] - \varepsilon\{a\nabla v \cdot n\}[u])\, ds$$
$$+ \sum_{E\in\tau_{h,m}} (b \cdot \nabla u, v) - \sum_{e\in\partial\tau_{h,m}} \int |b \cdot n|\{u\}[v]\, ds + \sigma \sum_{e\in\partial\tau_{h,m}} \int [u][\, v] ds. \qquad (5.3)$$

$$B_m(v) = \sum_{E\in\tau_{h,m}} (f, v) + \sum_{e\in\Gamma_N} \int u_N\, v ds - \sum_{e\in\Gamma_D} \int \varepsilon a\nabla v \cdot n u_D\, ds$$
$$+ \sum_{e\in\Gamma_D} \int |b \cdot n| u_D v\, ds - \sigma \sum_{e\in\Gamma_D} \int u_D v ds. \qquad (5.4)$$

The DGFEM is: find $u_h \in V_{h,\tau}$ such that:

$$\int_{I_m} (u_{h,t}, v)\, dt + ([u_h]_{m-1}, v_{m-1}^{+}) + \int_{I_m} A_m(u_h, v)\, dt = \int_{I_m} B_m(v)\, dt \qquad (5.5)$$

where,

$$A_m(u_h, v) = \sum_{E \in \tau_{h,m}} a(\nabla u_h, \nabla v) - \sum_{e \in \partial\tau_{h,m}} \int (\{a\nabla u_h \cdot n\}[v] - \varepsilon\{a\nabla v \cdot n\}[u_h])\, ds$$

$$+ \sum_{E \in \tau_{h,m}} (b \cdot \nabla u_h, v) - \sum_{e \in \partial\tau_{h,m}} \int |b \cdot n|\{u_h\}[v]\, ds + \sigma \sum_{e \in \partial\tau_{h,m}} \int [u_h][\,v] ds. \qquad (5.6)$$

$$B_m(v) = \sum_{E \in \tau_{h,m}} (f, v) + \sum_{e \in \Gamma_N} \int u_N\, v ds - \sum_{e \in \Gamma_D} \int \varepsilon a \nabla v \cdot n u_D\, ds$$

$$+ \sum_{e \in \Gamma_D} \int |b \cdot n| u_D v\, ds - \sigma \sum_{e \in \Gamma_D} \int u_D v ds. \qquad (5.7)$$

where,

$$V_{h,\tau} = \{v \in L^2(\Omega);\ v|_E \in P^k(E), \forall E \in \tau_{h,m}\}$$

and $P^k(E)$ = set of polynomials of degree at most k on E and $k \geq 1$ is an integer.

6. Properties of the bilinear form $A_m(u, v)$.

Let V be Hilbert space with scalar product $(\cdot,\cdot)_V$, $(V = H^1(\Omega))$, and corresponding norm $\|u\|_{H^1(\Omega)}$. suppose that $A_m(u, v)$ is bilinear form on $V \times V$. We prove the properties of the bilinear form (v- *elliptic* and *continuity*).

Lemma 1. (v−*elliptic*). Assume that the penalty value σ is sufficiently large and that, $\beta_0 \geq (d-1)^{-1}$, there exist a positive constant k independent of h and τ such that,

$$A_m(u\,, u) \geq k\|u\|^2_{H^1(\tau_{h,m})}, \quad \forall u \in V,\ \ \forall E \in \tau_{h,m}$$

Proof: put $v = u$ in equation (5.3), we get

$$A_m(u, u) = \sum_{E \in \tau_{h,m}} a(\nabla u, \nabla u) + (\varepsilon - 1) \sum_{e \in \partial\tau_{h,m}} \int \{a\nabla u \cdot n\}[u]\, ds + \sum_{E \in \tau_{h,m}} (b \cdot \nabla u, u)$$

$$-\sum_{e\in\partial\tau_{h,m}}\int|b\cdot n|\{u\}[u]\,ds\ +\sigma\sum_{e\in\partial\tau_{h,m}}\int[u]^2\,ds=\sum_{i=1}^{4}A_m^{(i)} \qquad (6.1)$$

Define the energy norm,

$$A_m(u\,,u)=\left(\sum_{E\in\tau_{h,m}}\left\|a^{\frac{1}{2}}\nabla u\right\|_{L^2(E)}^2+\sigma\sum_{e\in\partial\tau_{h,m}}\|[u]\|_{L^2(e)}^2\right)^{\frac{1}{2}}=\|u\|_{H^1(\tau_{h,m})}$$

To estimate $A_m^{(1)}$,

$$A_m^{(1)}=\sum_{E\in\tau_{h,m}}a(\nabla u,\nabla u)=\sum_{E\in\tau_{h,m}}\left\|a^{\frac{1}{2}}\nabla u\right\|_{L^2(E)}^2 \qquad (6.2)$$

for $A_m^{(2)}$, by Schwartz and young inequalities we have,

$$A_m^{(2)}=(\varepsilon-1)\sum_{e\in\partial\tau_{h,m}}\int\{a\nabla u\cdot n\}[u]ds$$

$$\le\sum_{e\in\partial\tau_{h,m}}\|\{a\,\nabla u\cdot n\}\|_{L^2(e)}\,\|[u]\|_{L^2(e)}$$

$$=(\varepsilon-1)\sigma^{\frac{1}{2}}\sigma^{-\frac{1}{2}}\sum_{e\in\partial\tau_{h,m}}\|\{a\,\nabla u\cdot n\}\|_{L^2(e)}\,\|[u]\|_{L^2(e)}$$

$$=(\varepsilon-1)\sigma^{-\frac{1}{2}}\sum_{e\in\partial\tau_{h,m}}\|\{a\,\nabla u\cdot n\}\|_{L^2(e)}\sigma^{\frac{1}{2}}\,\|[u]\|_{L^2(e)}$$

since,

$$\sigma^{-\frac{1}{2}}\|\{a\,\nabla u\cdot n\}\|_{L^2(e)}=\left|\frac{a}{2}\right|\sigma^{-\frac{1}{2}}\left\|(\nabla u\ \cdot n)_{E_1}+(\nabla u\ \cdot n)_{E_2}\right\|_{L^2(e)}$$

$$\le\left|\frac{a}{2}\right|\left(\frac{1}{h}\right)^{-\frac{1}{2}}\left\|(\nabla u\ \cdot n)_{E_1}\right\|_{L^2(e)}+\left\|(\nabla u\ \cdot n)_{E_2}\right\|_{L^2(e)}$$

where $E_1=E_+$ and $E_2=E_-$, from the trace inequality [17], we have,

$$\le\frac{|a|C_t}{2}h^{\frac{1}{2}}\left(h_{E_1}^{\frac{-1}{2}}\|\nabla u\|_{L^2(E_1)}+h_{E_2}^{\frac{-1}{2}}\|\nabla u\|_{L^2(E_2)}\right)$$

$$\leq \frac{|a|C_t}{2}\left(\|\nabla u\|_{L^2(E_1)} + \|\nabla u\|_{L^2(E_2)}\right)$$

$$\leq |a|C_t\|\nabla u\|_{L^2(E)} = C_t\|a\nabla u\|_{L^2(E)}$$

Where

$|e| \leq h_E^{d-1} \leq h^{d-1}$, $\forall e \in \partial E$, [17]. And C_t is a constant function [16]. Then,

$$A_m^{(2)} \leq (\varepsilon - 1) \sum_{e\in\partial\tau_{h,m}} C_t\|a\nabla u\|_{L^2(E)}\sigma^{\frac{1}{2}} \|[u]\|_{L^2(e)}$$

using young inequality we have,

$$\leq \frac{\mu}{2}\sum_{E\in\tau_{h,m}} \left\|a^{\frac{1}{2}}\nabla u\right\|_{L^2(E)}^2 + \frac{C_t^2(\varepsilon-1)^2}{2\mu}\sum_{e\in\partial\tau_{h,m}} \sigma \|[u]\|^2{}_{L^2(e)}$$

$$\leq \gamma\left(\sum_{E\in\tau_{h,m}} \left\|a^{\frac{1}{2}}\nabla u\right\|_{L^2(E)}^2 + \sigma\sum_{e\in\partial\tau_{h,m}} \|[u]\|^2{}_{L^2(e)}\right)$$

$$= \gamma\|u\|_{H^1(\tau_{h,m})}^2. \qquad (6.3)$$

To estimate $A_m^{(3)}$,

$$A_m^{(3)} = \sum_{E\in\tau_{h,m}} (b\cdot\nabla u, u) - \sum_{e\in\partial\tau_{h,m}} \int |b\cdot n|\{u\}[u]\, ds = A_m^{(31)} + A_m^{(32)} \qquad (6.4)$$

for $A^{(31)}$, by Schwartz and young inequality, we have

$$A_m^{(31)} = \sum_{E\in\tau_{h,m}} \int b\cdot \nabla u\, u\, dx \leq \sum_{E\in\tau_{h,m}} |b|\, \|\nabla u\|_{L^2(E)}\|u\|_{L^2(E)}$$

$$\leq \frac{\mu}{2}\|u\|_{L^2(\tau_{h,m})}^2 + \frac{b^2}{2\mu}\|\nabla u\|_{L^2(\tau_{h,m})}^2$$

$$\leq \gamma\left(\|u\|_{L^2(E)}^2 + \|\nabla u\|_{L^2(E)}^2\right) = \gamma\|u\|_{H^1(\tau_{h,m})}^2$$

hence,

$$A_m^{(31)} \leq \gamma \|u\|^2_{H^1(\tau_{h,m})}. \tag{6.5}$$

where, γ = m, ax$\left\{\frac{\mu}{2}, \frac{C_t{}^2(\varepsilon-1)^2}{2\mu}, \frac{b^2}{2\mu}\right\}$.

To estimate $A_m^{(32)}$,

$$A_m^{(32)} = -\sum_{e\in\partial\tau_{h,m}} \int |b\cdot n|[u]\,\{u\}\,ds \quad \leq \sum_{e\in\partial\tau_{h,m}} |b\cdot n|\|[u]\|_{L^2(e)}\|\,\{u\}\|_{L^2(e)}$$

$$= \sigma^{\frac{1}{2}-\frac{1}{2}} \sum_{e\in\partial\tau_{h,m}} |b\cdot n|\|[u]\|_{L^2(e)}\|\,\{u\}\|_{L^2(e)}$$

$$= \sum_{e\in\partial\tau_{h,m}} |b\cdot n|\sigma^{\frac{1}{2}}\|[u]\|_{L^2(e)}\sigma^{-\frac{1}{2}}\|\,\{u\}\|_{L^2(e)}$$

since,

$$\sigma^{-\frac{1}{2}}\|\,\{u\}\|_{L^2(e)} \leq \frac{1}{2}h^{\frac{1}{2}}\left(\left\|(u)_{E_1}\right\|_{L^2(e)} + \left\|(u)_{E_2}\right\|_{L^2(e)}\right)$$

from the trace and Poincare inequality we have,

$$\leq \frac{1}{2}h^{\frac{1}{2}}C_t\left(h_{E_1}^{-\frac{1}{2}}\left\|(u)_{E_1}\right\|_{L^2(E_1)} + h_{E_2}^{-\frac{1}{2}}\left\|(u)_{E_2}\right\|_{L^2(E_2)}\right)$$

$$= \frac{1}{2}C_t\left(\left\|(u)_{E_1}\right\|_{L^2(E_1)} + \left\|(u)_{E_2}\right\|_{L^2(E_2)}\right)$$

$$\leq \frac{1}{2}C_t\left(\|u\|_{L^2(E)} + \|u\|_{L^2(E)}\right) = C_t\|u\|_{L^2(\tau_{h,m})}$$

$$\leq C_t\|u\|_{H^1(\tau_{h,m})}.$$

then,

$$A_m^{(32)} \leq \sum_{e\in\partial\tau_{h,m}} C_t|b\cdot n|\sigma^{\frac{1}{2}}\|[u]\|_{L^2(e)}\ \|u\|_{H^1(E)}$$

$$\leq \delta \sum_{e\in\partial\tau_{h,m}} \sigma^{\frac{1}{2}}\|[u]\|_{L^2(e)}\ \|u\|_{H^1(\mathrm{E})}$$

$$\leq \frac{\mu}{2} \sum_{e\in\partial\tau_{h,m}} \sigma\|[u]\|^2{}_{L^2(e)} + \frac{\delta^2}{2\mu}\|u\|^2_{H^1(\tau_{h,m})}. \tag{6.6}$$

substituting (6.5)and (6.6) in (6.4) we have,

$$A_m^{(3)} \leq \gamma\|u\|^2_{H^1(\tau_{h,m})} + \frac{\mu}{2} \sum_{e\in\partial\tau_{h,m}} \sigma\|[u]\|^2{}_{L^2(e)} + \frac{\delta^2}{2\mu}\|u\|^2_{H^1(\tau_{h,m})}. \tag{6.7}$$

To estimate $A_m^{(4)}$,

$$A_m^{(4)} = \sigma \sum_{e\in\partial\tau_{h,m}} \int [u]^2\, ds \leq \sigma \sum_{e\in\partial\tau_{h,m}} \|[u]\|^2{}_{L^2(e)}. \tag{6.8}$$

substituting (6.2), (6.3), (6.7) and (6.8) in (6.1) we have,

$$A_m(u\,,u) = \sum_{E\in\tau_{h,m}} \left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \gamma\|u\|^2_{H^1(\tau_{h,m})} + \sigma \sum_{e\partial\tau_{h,m}} \|[u]\|^2{}_{L^2(e)}$$

$$+\gamma\|u\|^2_{H^1(\tau_{h,m})} + \frac{\mu}{2}\sigma \sum_{e\partial\tau_{h,m}} \|[u]\|^2{}_{L^2(e)} + \frac{\delta^2}{2\mu}\|u\|^2_{H^1(\tau_{h,m})}$$

$$= \sum_{E\in\tau_{h,m}} \left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \left(1+\frac{\mu}{2}\right)\sigma \sum_{e\partial\tau_{h,m}} \|[u]\|^2{}_{L^2(e)} + \left(2\gamma + \frac{\delta^2}{2\mu}\right)\|u\|^2_{H^1(\tau_{h,m})}$$

$$\geq \alpha\left(\sum_{E\in\tau_{h,m}} \left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \sigma \sum_{e\partial\tau_{h,m}} \|[u]\|^2{}_{L^2(e)}\right) + \lambda\|u\|^2_{H^1(\tau_{h,m})}$$

then,

$$A_m(u\,,u) \geq \mathrm{k}\|u\|^2_{H^1(\tau_{h,m})}\,.$$

where,

$\delta \geq C_t|b\cdot n|$, $\alpha = min\left\{1,\left(1+\frac{\mu}{2}\right)\right\}$, $\lambda \leq \left(2\gamma + \frac{\delta^2}{2\mu}\right)$, and $\mathrm{k} \leq (\alpha+\lambda)$.

Lemma 2.($countinuity$), a bilinear form defined on V space equipped with norm $\|\cdot\|_V$ is continuous if there is a positive constant ζ such that,

$$A_m(u\,,v) \le \zeta \|u\|_{H^1(\tau_{h,m})} \, \|v\|_{H^1(\tau_{h,m})}. \qquad \forall\, u,v\ \in V.$$

Proof: we introduce equation (5.3),

$$A_m(u,v) = \sum_{E\in\tau_{h,m}} a(\nabla u,\nabla v) + \sum_{E\in\tau_{h,m}} (b\cdot\nabla u,v) - \sum_{e\in\partial\tau_{h,m}} \int \{a\nabla u\cdot n\}[v]ds$$

$$+\varepsilon \sum_{e\in\partial\tau_{h,m}} \int \{a\nabla v\cdot n\}[u]ds - \sum_{e\in\partial\tau_{h,m}} \int |b\cdot n|\{u\}[v]ds \; + \sigma \sum_{e\in\partial\tau_{h,m}} \int [u][\,v]\,ds$$

$$= \sum_{i=1}^{5} A_m^{(i)}. \qquad (6.9)$$

To estimate $A_m^{(1)}$,

$$A_m^{(1)} = \sum_{E\in\tau_{h,m}} \big(a\,(\nabla u\;,\nabla v) + (b\cdot\nabla u\;,v)\big)$$

$$\le \sum_{E\in\tau_{h,m}} \big(|a|_{L^\infty}\|\nabla u\|_{L^2(E)}\|\nabla v\|_{L^2(E)} + |b|_{L^\infty}\|\nabla u\|_{L^2(E)}\|v\|_{L^2(E)}\big)$$

$$\le c \sum_{E\in\tau_{h,m}} \|\nabla u\|_{L^2(E)}\big(\|\nabla v\|_{L^2(E)} + \|v\|_{L^2(E)}\big)$$

$$\le c \sum_{E\in\tau_{h,m}} \big(\|\nabla u\|_{L^2(E)} + \|u\|_{L^2(E)}\big)\big(\|\nabla v\|_{L^2(E)} + \|v\|_{L^2(E)}\big)$$

$$= c\|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})}. \qquad (6.10)$$

where,

$$c = max\{|a|_{L^\infty}, |b|_{L^\infty}\}$$

To estimate $A_m^{(2)}$,

$$A_m^{(2)} = \sum_{e\in\partial\tau_{h,m}} \int \{a\nabla u\cdot n\}[v]\,ds$$

$$\le \sum_{e\in\partial\tau_{h,m}} \|\{a\,\nabla u\cdot n\}\|_{L^2(e)}\ \|[v]\|_{L^2(e)}$$

$$= \sigma^{\frac{1}{2}}\sigma^{-\frac{1}{2}} \sum_{e\in\partial\tau_{h,m}} \|\{a\,\nabla u\cdot n\}\|_{L^2(e)}\,\|[v]\|_{L^2(e)}$$

$$= \sigma^{-\frac{1}{2}} \sum_{e\in\partial\tau_{h,m}} \|\{a\,\nabla u\cdot n\}\|_{L^2(e)}\sigma^{\frac{1}{2}}\,\|[v]\|_{L^2(e)}$$

since,

$$\sigma^{-\frac{1}{2}}\|\{a\,\nabla u\cdot n\}\|_{L^2(e)} = \left|\frac{a}{2}\right|\sigma^{-\frac{1}{2}}\left\|(\nabla u\,\cdot n)_{E_1} + (\nabla u\,\cdot n)_{E_2}\right\|_{L^2(e)}$$

$$\leq \left|\frac{a}{2}\right|\left(\frac{1}{h}\right)^{-\frac{1}{2}}\left(\left\|(\nabla u\,\cdot n)_{E_1}\right\|_{L^2(e)} + \left\|(\nabla u\,\cdot n)_{E_2}\right\|_{L^2(e)}\right)$$

from the trace inequality we have,

$$\left|\frac{a}{2}\right| h^{\frac{1}{2}}\left(\left\|(\nabla u\,\cdot n)_{E_1}\right\|_{L^2(e)} + \left\|(\nabla u\,\cdot n)_{E_2}\right\|_{L^2(e)}\right) \leq \frac{|a|C_t}{2}\left(\|\nabla u\|_{L^2(E_1)} + \|\nabla u\|_{L^2(E_2)}\right)$$

$$\leq |a|C_t\|\nabla u\|_{L^2(E)} \tag{6.11}$$

similarly for,

$$\sigma^{\frac{1}{2}}\,\|[v]\|_{L^2(e)} = \sigma^{1-\frac{1}{2}}\,\|[v]\|_{L^2(e)} = \sigma^{-\frac{1}{2}}\,\sigma\|[v]\|_{L^2(e)} \leq \sigma C_t\|v\|_{L^2(E)} \tag{6.12}$$

from (6.11) and (6.12), we get,

$$A_m^{(2)} \leq |a|\sigma {C_t}^2 \sum_{e\in\partial\tau_h} \|\nabla u\|_{L^2(E)}\,\|v\|_{L^2(E)}$$

$$\leq |a|\sigma {C_t}^2 \sum_{e\in\partial\tau_h} \left(\|\nabla u\|_{L^2(E)} + \|u\|_{L^2(E)}\right)\left(\|v\|_{L^2(E)} + \|\nabla v\|_{L^2(E)}\right)$$

$$= |a|\sigma {C_t}^2\|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})}. \tag{6.13}$$

To estimate $A_m^{(3)}$,

$$A_m^{(3)} = -\varepsilon \sum_{e\in\partial\tau_{h,m}} \int \{a\nabla v\cdot n\}[u]\,ds \leq |a|\varepsilon\sigma {C_t}^2\|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})}. \tag{6.14}$$

similarly for $A_m^{(4)}$,

$$A_m^{(4)} = \sum_{e\in\partial\tau_{h,m}} \int |b\cdot n|\{u\}[v]ds \le \sigma {C_t}^2 \|u\|_{H^1(\tau_{h,m})} \|v\|_{H^1(\tau_{h,m})}. \qquad (6.15)$$

To estimate $A_m^{(5)}$,

$$A_m^{(5)} = \sigma \sum_{e\in\partial\tau_{h,m}} \int [u][\,v]ds \le \sigma \sum_{e\in\partial\tau_{h,m}} \|[u]\|_{L^2(e)} \|[v]\|_{L^2(e)}$$

$$= \sigma^2 \sum_{e\in\partial\tau_h} \sigma^{-\frac{1}{2}} \|[u]\|_{L^2(e)} \sigma^{-\frac{1}{2}} \|[v]\|_{L^2(e)}.$$

since,

$\sigma^{-\frac{1}{2}}\|[u]\|_{L^2(e)} \le C_t \|u\|_{L^2(E)}$ and $\sigma^{-\frac{1}{2}}\|[v]\|_{L^2(e)} \le C_t \|v\|_{L^2(E)}$

then,

$$A_m^{(5)} \le \sigma^2 {C_t}^2 \|u\|_{L^2(\tau_{h,m})} \|v\|_{L^2(\tau_{h,m})} \le \sigma^2 {C_t}^2 \|u\|_{H^1(\tau_{h,m})} \|v\|_{H^1(\tau_{h,m})}. \qquad (6.16)$$

substituting (6.10), (6.13), (6.14), (6.15) and (6.16) in (6.9), we have

$$A_m(u,v) = \sum_{E\in\tau_{h,m}} a(\nabla u,\nabla v) + \sum_E (b\cdot\nabla u, v) - \sum_{e\in\partial\tau_{h,m}} \int \{a\nabla u\cdot n\}[v]ds$$

$$+\varepsilon \sum_{e\in\partial\tau_{h,m}} \int \{a\nabla v\cdot n\}[u]ds - \sum_{e\in\partial\tau_{h,m}} \int |b\cdot n|\{u\}[v]ds + \sigma \sum_{e\in\partial\tau_{h,m}} \int [u][\,v]\,ds$$

$$\le c\|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})} - |a|\sigma {C_t}^2 \|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})}$$

$$+\varepsilon|a|\sigma {C_t}^2 \|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})} - \sigma {C_t}^2 \|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})}$$

$$+\sigma^2 {C_t}^2 \|u\|_{H^1(E)}\|v\|_{H^1(\tau_{h,m})}$$

$$= \left(c + |a|\sigma {C_t}^2(\varepsilon - 1) + \sigma {C_t}^2(\sigma - 1)\right) \|u\|_{H^1(\tau_{h,m})}\|v\|_{H^1(\tau_{h,m})}.$$

then,

$$A_m(u\,,v) \le \zeta \|u\|_{H^1(\tau_{h,m})}\,\|v\|_{H^1(\tau_{h,m})}.$$

where,

$$\zeta \ge \left(c + |a|\sigma {C_t}^2(\varepsilon - 1) + \sigma {C_t}^2(\sigma - 1)\right)$$

Lemma 3.(*stability*): there exist a constant $\alpha > 0$ independent of h and τ such that:

$$\|(u_h)_m^-\|^2{}_{L^2(\tau_{h,m})} + \|[u_h]_{m-1}\|^2{}_{L^2(\tau_{h,m})} + \delta\|u_h\|^2_{L^2(I_m;\,H^1(\tau_{h,m}))}$$

$$\le \alpha \int_{I_m} \left(\|f\|^2_{L^2(\tau_{h,m})} + \sum_{e\in\partial\tau_{h,m}} \|u_D\|^2_{L^2(\Gamma_D)} + \sum_{e\in\partial\tau_{h,m}} \|u_N\|^2_{L^2(\Gamma_N)}\right) dt.$$

Proof: choose $v = u_h$ in equation (5.5), we get,

$$\int_{I_m} (u_{h,t}, u_h)\,dt + ([u_h]_{m-1}, (u_h)_{m-1}^+) + \int_{I_m} A_m(u_h, u_h)\,dt$$

$$= \int_{I_m} \left(\sum_{E\in\tau_{h,m}} (f, u_h) + \sum_{e\in\Gamma_N} \int u_N u_h ds - \sum_{e\in\Gamma_D} \int \varepsilon a \nabla u_h \cdot n u_D\,ds \right.$$

$$\left. + \sum_{e\in\Gamma_D} \int |b\cdot n| u_D u_h\,ds \quad -\sigma \sum_{e\in\Gamma_D} \int u_D u_h ds \right) dt \;=\; \int_{I_m} \sum_{i=1}^{5} L_m^{(i)}\,dt \qquad (6.17)$$

to estimate the first term in the lift hand side,

$$\int_{I_m} (u_{h,t}, u_h)\,dt = \frac{1}{2}\int_{I_m} \frac{d}{dt}\|u_h\|^2_{L^2(\tau_{h,m})}\,dt = \|(u_h)_m^-\|^2{}_{L^2(\tau_{h,m})}$$

$$-\|(u_h)_{m-1}^+\|^2{}_{L^2(\tau_{h,m})} \qquad (6.18)$$

to estimate second term,

$$([u_h]_{m-1}, (u_h)_{m-1}^+) \le \|[u_h]_{m-1}\|_{L^2(\tau_{h,m})}\|(u_h)_{m-1}^+\|_{L^2(\tau_{h,m})}$$

$$\leq \|[u_h]_{m-1}\|^2{}_{L^2(\tau_{h,m})} + \|(u_h)^+_{m-1}\|^2{}_{L^2(\tau_{h,m})} \qquad (6.19)$$

from (6.18) and (6.19) we have,

$$\int_{I_m} (u_{h,t}, u_h)\, dt + ([u_h]_{m-1}, (u_h)^+_{m-1}) = \|(u_h)^-_m\|^2{}_{L^2(\tau_{h,m})}$$

$$+\|[u_h]_{m-1}\|^2{}_{L^2(\tau_{h,m})} \qquad (6.20)$$

for the third term, from lemma (4), we have,

$$\int_{I_m} A_m(u_h, u_h)\, dt \geq k \int_{I_m} \|u_h\|^2_{H^1(\tau_{h,m})}\, dt = k\|u_h\|^2_{L^2(I_m;\, H^1(\tau_{h,m}))} \qquad (6.21)$$

To estimate $L_m^{(1)}$, by young's inequality, we have,

$$L_m^{(1)} = (f, u_h) \leq \frac{\mu}{2}\|f\|^2_{L^2(\tau_{h,m})} + \frac{1}{2\mu}\|u_h\|^2_{L^2(\tau_{h,m})}$$

$$\leq C\left(\|f\|^2_{L^2(\tau_{h,m})} + \|u_h\|^2_{L^2(\tau_{h,m})}\right) \qquad (6.22).$$

To estimate $L_m^{(2)}$,

$$L_m^{(2)} = -\varepsilon \sum_{e\in\Gamma_D} \int a\nabla u_h \cdot n\, u_D ds \leq \varepsilon \sum_{e\in\partial\tau_{h,m}} \|a\nabla u_h \cdot n\|_{L^2(e)} \|u_D\|_{L^2(e)}$$

$$\leq \sum_{e\in\partial\tau_{h,m}} \left(\frac{\mu}{2}\|\nabla u_h\|^2_{L^2(e)} + \frac{\varepsilon^2}{2\mu}\|u_D\|^2_{L^2(\Gamma_D)}\right)$$

$$\leq C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)}\right) \qquad (6.23)$$

similarly for the terms,

$$L_m^{(3)} = \sum_{e\in\Gamma_N} \int u_N\, u_h ds \leq C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_N\|^2_{L^2(\Gamma_N)}\right). \qquad (6.24)$$

$$L_m^{(4)} = \sum_{e\in\Gamma_D} \int |b\cdot n| u_D\, u_h\, ds \;\le C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)} \right). \qquad (6.25)$$

$$L_m^{(5)} = -\sum_{e\in\Gamma_D} \sigma \int u_D\, u_h\, ds \le\; C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)} \right). \qquad (6.26)$$

substituting (6.18), (6.19), (6.20), (6.21), (6.22),(6.23), (6.24), (6.25) and (6.26) in (6.17), we have,

$$\|(u_h)_m^-\|^2{}_{L^2(\tau_{h,m})} + \|[u_h]_{m-1}\|^2{}_{L^2(\tau_{h,m})} + k\|u_h\|^2_{L^2(I_m;\,H^1(\tau_{h,m}))}$$

$$\le \int_{I_m} \Bigg(C\left(\|f\|^2_{L^2(\tau_{h,m})} + \|u_h\|^2_{L^2(\tau_{h,m})} \right) + C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)} \right)$$

$$+C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_N\|^2_{L^2(\Gamma_N)} \right) + C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)} \right).$$

$$+C \sum_{e\in\partial\tau_{h,m}} \left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)} \right) \Bigg)\, dt$$

re arrangement above inequality, we have,

$$\|(u_h)_m^-\|^2{}_{L^2(\tau_{h,m})} + \|[u_h]_{m-1}\|^2{}_{L^2(\tau_{h,m})} + k\|u_h\|^2_{L^2(I_m;\,H^1(E))}$$

$$\le C \int_{I_m} \Bigg(\|f\|^2_{L^2(\tau_{h,m})} + \|u_h\|^2_{H^1(\tau_{h,m})} + 4\|u_h\|^2_{H^1(\tau_{h,m})} + \sum_{e\in\partial\tau_{h,m}} 3\|u_D\|^2_{L^2(\Gamma_D)}$$

$$+ \sum_{e\in\partial\tau_{h,m}} \|u_N\|^2_{L^2(\Gamma_N)} \Bigg)\, dt.$$

$$\|(u_h)_m^-\|^2{}_{L^2(\tau_{h,m})} + \|[u_h]_{m-1}\|^2{}_{L^2(\tau_{h,m})} + (k-5C)\|u_h\|^2_{L^2(I_m;\,H^1(\tau_{h,m}))}$$

$$\le \alpha \int_{I_m} \left(\|f\|^2_{L^2(\tau_{h,m})} + \sum_{e\in\partial\tau_{h,m}} \|u_D\|^2_{L^2(\Gamma_D)} + \sum_{e\in\partial\tau_{h,m}} \|u_N\|^2_{L^2(\Gamma_N)} \right) dt.$$

hence,

$$\|(u_h)_m^-\|^2_{L^2(\tau_{h,m})} + \|[u_h]_{m-1}\|^2_{L^2(\tau_{h,m})} + \delta\|u_h\|^2_{L^2(I_m;\,H^1(\tau_{h,m}))}$$

$$\leq \alpha \int_{I_m} \left(\|f\|^2_{L^2(\tau_{h,m})} + \sum_{e\in\partial\tau_{h,m}} \|u_D\|^2_{L^2(\Gamma_D)} + \sum_{e\in\partial\tau_{h,m}} \|u_N\|^2_{L^2(\Gamma_N)}\right) dt.$$

where, $C = max\left\{\frac{\mu}{2}, \frac{\varepsilon^2}{2\mu}\right\}$, $\delta \leq (k - 5C)$ and $\alpha \geq 3C$

7. The error estimate.

Theorem (1): suppose that $u \in L^2(I_m;\, H^r(\Omega))$ and that u_0 belongs to $H^r(\Omega)$ and let σ sufficiently large then there exist a constant C such that :

$$\|u - u_h\|_{L^2(I_m, L^2(\tau_{h,m}))} \leq C(h^{r+1} + \tau^{s+1}), \quad r, s \geq 1, \quad r \neq s.$$

Proof: let $\tilde{u}$ be the L^2 projection, and $e = u - u_h = u - \tilde{u} + \tilde{u} - u_h = \rho - \theta$, hen,

$$\|u - u_h\|_{L^2(I_m, L^2(\tau_{h,m}))} \leq \|u - \tilde{u}\|_{L^2(I_m, L^2(\tau_{h,m}))} + \|u_h - \tilde{u}\|_{L^2(I_m, L^2(\tau_{h,m}))}$$

$$= \|\rho\|_{L^2(I_m, L^2(\tau_{h,m}))} + \|\theta\|_{L^2(I_m, L^2(\tau_{h,m}))} \qquad (7.1)$$

from [19], we have,

$$\|\rho\|^2_{L^2(I_m, L^2(\tau_{h,m}))}$$

$$\leq c\left(h^{2r+2}|u|^2_{L^2\left(I_m, H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2}|u|^2_{H^{s+1}\left(I_m, L^2(\tau_{h,m})\right)}\right).$$

hence,

$$\|\rho\|_{L^2(I_m, L^2(\tau_{h,m}))} \leq \sqrt{c}(h^{r+1} + \tau^{s+1}) \qquad (7.2)$$

by subtracting (5.5) from (5.2) we have,

$$\int_{I_m} \left((u - u_h)_t, v) + A_m(u - u_h, v)\right) dt + ([u - u_h]_{m-1}, v^+_{m-1})$$

$$= \int_{I_m} \left((\rho - \theta)_t, v) + A_m(\rho - \theta, v)\right) dt + ([\rho - \theta]_{m-1}, v^+_{m-1}) = 0.$$

then,

$$\int_{I_m} \left((\theta_t, v) + A_m(\theta, v)\right) dt + ([\theta]_{m-1}, v_{m-1}^+) = \int_{I_m} \left((\rho_t, v) + A_m(\rho, v)\right) dt$$
$$+([\rho]_{m-1}, v_{m-1}^+).$$

for bound θ, let $v = \theta$, we have,

$$\int_{I_m} (\theta_t, \theta)\, dt + ([\theta]_{m-1}, \theta_{m-1}^+) + \int_{I_m} A_m(\theta, \theta)\, dt$$
$$= \int_{I_m} (\rho_t, \theta)\, dt + ([\rho]_{m-1}, \theta_{m-1}^+) + \int_{I_m} A_m(\rho, \theta)\, dt. \tag{7.3}$$

since,

$$\int_{I_m} (\theta_t, \theta)\, dt = \frac{1}{2} \int_{I_m} \frac{d}{dt} \|\theta\|^2{}_{L^2(\tau_{h,m})}\, dt = \|\theta_m^-\|_{L^2(\tau_{h,m})}^2 - \|\theta_{m-1}^+\|_{L^2(\tau_{h,m})}^2 \tag{7.4}$$

$$([\theta]_{m-1}, \theta_{m-1}^+) \le \|[\theta]_{m-1}\|_{L^2(\tau_{h,m})} \|\theta_{m-1}^+\|_{L^2(\tau_{h,m})}$$
$$\le \frac{1}{2}\left(\|[\theta]_{m-1}\|^2{}_{L^2(\tau_{h,m})} + \|\theta_{m-1}^+\|^2{}_{L^2(\tau_{h,m})}\right) \tag{7.5}$$

from (7.4) and (7.5), we have,

$$\int_{I_m} (\theta_t, \theta)\, dt + ([\theta]_{m-1}, \theta_{m-1}^+) = \|\theta_m^-\|_{L^2(\tau_{h,m})}^2 + \frac{1}{2}(\|[\theta]_{m-1}\|^2{}_{L^2(\tau_{h,m})}$$
$$-\|\theta_{m-1}^+\|_{L^2(\tau_{h,m})}^2) \tag{7.6}$$

from lemma (1) we have,

$$\int_{I_m} A_m(\theta, \theta)\, dt \ge \mathrm{k} \int_{I_m} \|\theta\|_{H^1(\tau_{h,m})}^2\, dt = \mathrm{k}\|\theta\|_{L^2(I_m, H^1(\tau_{h,m}))}^2. \tag{7.7}$$

clearly,

$$\int_{I_m} (\rho_t, \theta)\, dt = (\rho_m^-, \theta_m^-) - (\rho_{m-1}^+, \theta_{m-1}^+) - \int_{I_m} (\rho, \theta_t)\, dt$$

and,

$$\int_{I_m} (\rho, \theta_t)\, dt \; = 0 \;\text{ and }\; (\rho_m^-, \theta_m^-) = 0,$$

then we have,

$$\int_{I_m} (\rho_t, \theta)\, dt + ([\rho]_{m-1}, \theta_{m-1}^+) = -(\rho_{m-1}^+, \theta_{m-1}^+) + (\rho_{m-1}^+, \theta_{m-1}^+) + (\rho_{m-1}^-, \theta_{m-1}^+)$$

$$= (\rho_{m-1}^-, \theta_{m-1}^+) \le \frac{1}{2}\left(\|\rho_{m-1}^-\|^2{}_{L^2(\tau_{h,m})} + \|\theta_{m-1}^+\|^2{}_{L^2(\tau_{h,m})}\right). \qquad (7.8)$$

from lemma (2), we have,

$$A_m(\rho, \theta) \le \zeta \|\rho\|_{H^1(\tau_{h,m})} \|\theta\|_{H^1(\tau_{h,m})},$$

then,

$$\int_{I_m} A_m(\rho, \theta)\, dt \le \zeta \int_{I_m} \|\rho\|_{H^1(\tau_{h,m})} \|\theta\|_{H^1(\tau_{h,m})}\, dt$$

$$\le \frac{\epsilon\zeta_1}{2} \int_{I_m} \|\rho\|^2{}_{H^1(\tau_{h,m})}\, dt + \frac{\zeta_1}{2\epsilon} \int_{I_m} \|\theta\|^2{}_{H^1(\tau_{h,m})}\, dt$$

$$= \frac{\epsilon\zeta_1}{2} \|\rho\|^2_{L^2(I_m, H^1(\tau_{h,m}))} + \frac{\zeta_1}{2\epsilon} \|\theta\|^2_{L^2(I_m, H^1(\tau_{h,m}))}.$$

for the first term,

$$\frac{\epsilon\zeta_1}{2} \|\rho\|^2_{L^2(I_m, H^1(\tau_{h,m}))}$$

$$\le c\left(h^{2r+2} |u|^2_{L^2\left(I_m, H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2} |u|^2_{H^{s+1}\left(I_m, L^2(\tau_{h,m})\right)}\right).$$

dividing by $\frac{\epsilon\zeta_1}{2}$, we have,

$$\|\rho\|^2_{L^2(I_m, H^1(\tau_{h,m}))} \le c_1\left(h^{2r+2} |u|^2_{L^2\left(I_m, H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2} |u|^2_{H^{s+1}\left(I_m, L^2(\tau_{h,m})\right)}\right).$$

hence,

$$\int_{I_m} A_m(\rho,\theta)\,dt \le c_1\left(h^{2r+2}|u|^2_{L^2\left(I_m,H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2}|u|^2_{H^{s+1}\left(I_m,L^2(\tau_{h,m})\right)}\right)$$

$$+ \frac{\zeta_1}{2\epsilon}\|\theta\|^2_{L^2(I_m,H^1(\tau_{h,m}))}. \qquad (7.9)$$

by substituting (7.6), (7.7), (7.8) and (7.9) in (7.3), we have,

$$\|\theta_m^-\|^2_{L^2(\tau_{h,m})} + \frac{1}{2}\left(\|[\theta]_{m-1}\|^2{}_{L^2(\tau_{h,m})} - 2\|\theta_{m-1}^+\|^2_{L^2(\tau_{h,m})}\right) + \mathrm{k}\|\theta\|^2_{L^2(I_m,H^1(\tau_{h,m}))}$$

$$\le c_1\left(h^{2r+2}|u|^2_{L^2\left(I_m,H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2}|u|^2_{H^{s+1}\left(I_m,L^2(\tau_{h,m})\right)}\right) + \frac{\zeta_1}{2\epsilon}\|\theta\|^2_{L^2(I_m,H^1(\tau_{h,m}))}$$

$$+ \frac{1}{2}\|\rho_{m-1}^-\|^2_{L^2(\tau_{h,m})}.$$

$$\mathrm{k}\|\theta\|^2_{L^2(I_m,H^1(\tau_{h,m}))} \le c_1\left(h^{2r+2}|u|^2_{L^2\left(I_m,H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2}|u|^2_{H^{s+1}\left(I_m,L^2(\tau_{h,m})\right)}\right)$$

$$+ \frac{\zeta_1}{2\epsilon}\|\theta\|^2{}_{L^2(I_m,H^1(\tau_{h,m}))}.$$

re arrangement above inequality, we have,

$$\left(\mathrm{k} - \frac{\zeta_1}{2\epsilon}\right)\|\theta\|^2_{L^2(I_m,H^1(\tau_{h,m}))}$$

$$\le c_1\left(h^{2r+2}|u|^2_{L^2\left(I_m,H^{r+1}(\tau_{h,m})\right)} + \tau^{2s+2}|u|^2_{H^{s+1}\left(I_m,L^2(\tau_{h,m})\right)}\right).$$

hence,

$$\|\theta\|^2_{L^2(I_m,H^1(\tau_{h,m}))} \le \beta\left(h^{2r+2}|u|^2_{L^2\left(I_m,H^1(\tau_{h,m})\right)} + \tau^{2s+2}|u|^2_{H^1\left(I_m,L^2(\tau_{h,m})\right)}\right)$$

$$\le \beta(h^{2r+2} + \tau^{2s+2}).$$

hence,

$$\|\theta\|_{L^2(I_m,H^1(\tau_{h,m}))} \le \sqrt{\beta}(h^{r+1} + \tau^{s+1}) \qquad (7.10)$$

substituting (7.2) and (7.10) in (7.1), we have,

$$\|u - u_h\|^2_{L^2(I_m, L^2(\tau_{h,m}))} \leq C(h^{r+1} + \tau^{s+1}).$$

where,

$$c_1 = \frac{2c}{\epsilon\zeta_1}, \beta \geq \frac{c_1}{\left(\mathrm{k} - \frac{\zeta_1}{2\epsilon}\right)}, C = max\{\sqrt{\beta}, \sqrt{c}\} \text{ and } \mathrm{k} \neq \frac{\zeta_1}{2\epsilon}$$

8.Conclusions.

This paper is devoted to the theoretical analysis of error estimates of the space-time DGFE method for linear convection- diffusion problem. The DGFE method is applied separately in space and time using, in general, different space grids on different time levels. There are three versions depend on the choices of the parameters ε and σ.

• If $\varepsilon = -1$ and σ is bounded below by a large enough constant, the resulting method is called the symmetric interior penalty Galerkin (SIPG) method.

• If $\varepsilon = +1$ and $\sigma = 1$, the resulting method is called the non-symmetric interior penalty Galerkin (NIPG) method.

• If $\varepsilon = 0$ and $\sigma > 0$ we obtain the incomplete interior penalty Galerkin (IIPG) method.

We prove the properties of the bilinear form $A(u, v)$, (v-elliptic and continuity) of DG, stability and proved the approximate solution is converges with error of order $O(h^{r+1} + \tau^{s+1})$.

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