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SOLUTION OF NONLINEAR INITIAL-VALUE PROBLEMS BY THE ALTERNATING DIRECTION IMPLICIT FORMULATION OF THE DIFFERENTIAL QUADRATURE METHOD

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Abstract

The alternating direction implicit formulation of the differential quadrature method [1, 2, 3 and 4] is applied to the solution of nonlinear initial -value problems. The weighting coefficients are computing by Fourier series expansion. Numerical results of two examples are show that the present method with Fourier series expansion has a high accuracy and good convergence comparing with using Lagrange interpolation polynomial to compute the weighting coefficients.

INTRODUCTION:

Consider the two-dimensional Burger equation:

$$\frac{\partial u_1}{\partial t}+\alpha u_1\frac{\partial u_1}{\partial x}+\alpha u_2\frac{\partial u_1}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)=0 \quad (x,y)\in\Omega\ ,\ \ \mathrm{t}>0\ \ (1.1a)$$

$$\frac{\partial u_2}{\partial t}+\alpha u_1\frac{\partial u_2}{\partial x}+\alpha u_2\frac{\partial u_2}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right)=0 \quad (x,y)\in\Omega\ ,\ \ \mathrm{t}>0\ \ (1.1b)$$

The computational domain is taken as $\Omega=\{(x,y):0\le x,y\le L\}$ with initial conditions

$$u_1(x,y,0)=\emptyset_1(x,y) \quad , \quad u_2(x,y,0)=\emptyset_2(x,y) \qquad (1.1c)$$

And the boundary conditions

$$\left.\begin{array}{l} u_1(x,y,t)=f(x,y,t) \\ u_2(x,y,t)=g\,(x,y,t)\end{array}\right\}\ (x,y)\in\partial\,\Omega\ ,\quad \mathrm{t}>0 \qquad (1.1\mathrm{d})$$

Where Re is the Reynolds number, α constant, $u_1$and u_2 are velocity components and $\emptyset_1,\emptyset_2,f$ and g are the known functions. For a positive integer$n$, let $h={}^{L}/_{n}$ denote the step size of spatial space and Δt is the step size with respect to time. Burgers' equation is a fundamental nonlinear partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of dynamics, heat conduction, shock waves, and acoustic waves [6, 10]. This equation is a special form of incompressible Navier-Stokes equation without having pressure term and continuity equation. The first attempt to solve Burgers' equation analytically was given by Bateman, who derived the steady solution for a simple one- dimensional Burgers' equation, which was used by Burger to model turbulence. In the past several years, numerical solution to one-dimensional Burgers' equation and system of multidimensional Burgers' equations have attracted a lot of attention the researchers [12]. The development of new techniques from the standpoint of computational efficiency and numerical accuracy is of primal interest. Since it has been developed, several researchers have applied successfully the differential quadrature method to solve a variety of problems in different fields of science and engineering. In this work, we applied a new technique of the differential quadrature method [1,2,3,4] for solving boundary value problems. With this technique, we use Fourier series expansion to compute the weighting coefficients. Our results are comparing with [2] that is used Lagrange interpolation polynomial to compute the weighting coefficients, and show that the present method has a high accuracy; good convergence.

2. Differential Quadrature Method

The differential quadrature is a numerical technique used to solve the initial and boundary value problems. This method was proposed by Bellman in the early 70s . [5]. The essence of the method is that the partial (ordinary) derivatives of a function with respect to a variable in governing equation are approximated by a weighted linear sum of function values at all discrete points in that direction, then the equation can be transformed into a set of ordinary differential equations or algebraic equations. According to the DQM, The r^{th} -order partial derivatives $\frac{\partial^r u_1}{\partial x^r}$ of a function $u_1(x,y)$ at a point (x_i, y_j) and the s^{th} -order partial derivatives $\frac{\partial^s u_1}{\partial y^s}$ of a function $u_1(x,y)$ at a point (x_i, y_j) , can be approximated by the same formula given in [11],as:

$$\left.\frac{\partial^r u_1}{\partial x^r}\right|_{x=x_i} = \sum_{k=1}^{N} A_{ik}^{(r)} u_1(x_k, y) \quad , \quad i = 1,2, \dots, N \qquad (2.1)$$

$$\left.\frac{\partial^s u_1}{\partial y^s}\right|_{y=y_j} = \sum_{l=1}^{M} B_{jl}^{(s)} u_1(x, y_l) \quad , \quad j = 1,2, \dots, M \qquad (2.2)$$

where $A_{ik}^{(r)}, B_{jl}^{(s)}$ are the respective weighting coefficients for the r^{th} -order and s^{th}-order derivatives with respect to x and y respectively. Bellman et al. [5]

proposed two approaches to compute the weighting coefficients$A_{ik}^{(r)}$, $B_{jl}^{(s)}$. To improve Bellman's approaches in computing the weighting coefficients (WCs), many attempts have been made by researchers. One of the most attempts is introduced by Quan and Chang [8, 9]. After that, Shu's [11] introduced a general approach, which was inspired from Bellman's approach, was made available in the literature. Shu's recurrence formulation for higher order derivatives is given by Shu's [11] as:

$$A_{ik}^{(r)} = r\left(A_{ii}^{(r-1)}A_{ik}^{(1)} - \frac{A_{ik}^{(r-1)}}{\sin[(x_i - x_k)/2]}\right), k, i = 1, \dots, N, \quad 2 \le r \le N-1 \ , i \ne k \quad (2.3)$$

and

$$A_{ii}^{(r)} = -\sum_{k=1}^{N} A_{ik}^{(r)} \quad ,1 \le r \le N-1 \quad , \ i \ne k \quad ,i = 1,2, \dots, N \qquad (2.4)$$

where $A_{ik}^{(1)}$ are the weighting coefficients of the first order derivative given below

$$A_{ik}^{(1)} = \frac{M^{(1)}(x_i)}{\sin[(x_i - x_k)/2]M^{(1)}(x_k)} \qquad for \quad i \ne k$$

where $M(x) = \prod_{k=1}^{N} \sin[(x - x_k)/2]$ *and* $M^{(1)}(x_i) = \prod_{k=1,k\ne i}^{N} \sin[(x_i - x_k)/2]$

By differential quadrature method, we approximate the partial derivatives of the equation (1.1a). Using equations (2.1) and (2.2) in Burger equation (1.1a), we obtain the system of ordinary differential equations as:

$$\left.\frac{\partial u_1}{\partial t}\right|_{ij}^{n} + \sum_{k=1}^{N} \alpha {u_1}_{ij} A_{ik}^{(1)} {u_1}_{kj} + \sum_{l=1}^{M} \alpha {u_2}_{ij} B_{jl}^{(1)} {u_1}_{il} = \frac{1}{Re}\left(\sum_{k=1}^{N} A_{ik}^{(2)} {u_1}_{kj} + \sum_{l=1}^{M} B_{jl}^{(2)} {u_1}_{il}\right) \qquad (2.5)$$

Approximation the first-order derivatives with respect to the temporal variable in the equation (2.5) by using the forward differences and arrangement the terms of equation (2.5) ,we obtain the system of algebraic equation as:

$$\frac{u_{1_{ij}}^{n+1}-u_{1_{ij}}^{n}}{\Delta t}+\sum_{k=1}^{N}\left(\alpha u_{1_{ij}}^{n}A_{ik}^{(1)}-\frac{1}{Re}A_{ik}^{(2)}\right)u_{1_{kj}}^{n}+\sum_{l=1}^{M}(\alpha u_{2_{ij}}^{n}B_{jl}^{(1)}-\frac{1}{Re}B_{jl}^{(2)})u_{1_{il}}^{n}=0 \qquad (2.6)$$

3. Alternating Direction Technique of the DQM

The alternating direction implicit technique was introduced in the mid-50s by Peaceman and Rachford [7] for solving equations, which result from finite difference discretization of partial differential equations (PDEs). From iterative method's perspective, ADI method can be considered as special relaxation method, where a big system is simplified into a number of smaller systems such that each of them can be solved efficiently and the solution of the whole system is got from the solutions of the sub-systems in an iterative way. Using alternating direction implicit method into equation (2.6), we get the systems of algebraic equations in the form:

$$\frac{u_{1_{ij}}^{n+\frac{1}{2}}-u_{1_{ij}}^{n}}{\frac{\Delta t}{2}}+\sum_{k=1}^{N}\left(\alpha u_{1_{ij}}^{n}A_{ik}^{(1)}-\frac{1}{Re}A_{ik}^{(2)}\right)u_{1_{kj}}^{n+\frac{1}{2}}+\sum_{l=1}^{M}(\alpha u_{2_{ij}}^{n}B_{jl}^{(1)}-\frac{1}{Re}B_{jl}^{(2)})u_{1_{il}}^{n}=0 \qquad (3.1)$$

$$\frac{u_{1_{ij}}^{n+1}-u_{1_{ij}}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}+\sum_{k=1}^{N}\left(\alpha u_{1_{ij}}^{n+\frac{1}{2}}A_{ik}^{(1)}-\frac{1}{Re}A_{ik}^{(2)}\right)u_{1_{kj}}^{n+\frac{1}{2}}+\sum_{l=1}^{M}(\alpha u_{2_{ij}}^{n+\frac{1}{2}}B_{jl}^{(1)}-\frac{1}{Re}B_{jl}^{(2)})u_{1_{il}}^{n+1}=0 \qquad (3.2)$$

Formula (3.1) is used to compute the function values at all interval mesh points along rows and known as horizontal traverse or x −sweep. While, Formula (3.2) is used to compute function values at all interval mesh points along columns and known as vertical traverse or y −sweep.

In the same procedure, we approximate the partial derivatives of Equation (1.1b) by using ADI-DQM to obtain the systems of algebraic equations.

4. Numerical Experiments and Discussion

In this section, we apply ADI-DQM on two test problems which are also considered by other researchers. Problem 1. ([2])

We consider Burger equation (1.1) with $\alpha = 1,\ Re = 100, 150, 200\ ,\ L = 1$ and initial conditions in the following form:

$$u_1(x,y,0) = \frac{3}{4} - \frac{1}{4(1+\mathrm{e}^{\mathrm{Re}(y-x)/8})}\ , u_2(x,y,0) = \frac{3}{4} + \frac{1}{4(1+\mathrm{e}^{\mathrm{Re}(y-x)/8})} \qquad (4.1)$$

The exact solution is given by

$$u_1(x,y,t) = \frac{3}{4} - \frac{1}{4(1+\mathrm{e}^{\frac{\mathrm{Re}(4y-4x-t)}{32}})}, u_2(x,y,t) = \frac{3}{4} + \frac{1}{4(1+\mathrm{e}^{\frac{\mathrm{Re}(4y-4x-t)}{32}})} \quad (4.2)$$

The boundary conditions can be obtained easily from (4.2) by using $x, y = 0, 1$. In this problem, we found numerical results for $u_1$and u_2 and using equally spaced grid points. In Tables 1,2 and 3 we show the errors obtained in solving problem 1, at t=0.01, $\Delta t = 0.001$, $Re = 100, 150, 200$ and $x, y \in [0, 1]$ for different values of h, with the ADI-DQM by using Fourier series expansion and Lagrange interpolated polynomials to compute the weighted coefficients respectively, and also in Tables 4, 5 and 6 we show the errors obtained in solving problem 1 with the DQM by using Fourier series expansion and Lagrange interpolated polynomials to compute the weighted coefficients respectively. The numerical results given in tables 1,…,6 are confirm that the ADI-DQM and DQM by using Fourier series expansion to compute the weighting coefficients have a high accuracy, good convergence compare with using Lagrange interpolated polynomials to compute the weighting coefficients.

Table 1. Errors obtained by ADI-DQM for problem 1 with t=0.01 , $\Delta t = 0.001$ and $Re = 100$

$N \times M$	*Errors of u_1 -computing WCs by*		*Errors of u_2-computing WCs by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	1.106499E-06	2.472297E-07	7.943198E-06	3.380972E-06
10×10	8.206030E-07	2.301884E-07	7.616246E-06	3.019122E-06
15×15	5.852642E-07	2.748078E-08	8.575387E-06	3.813533E-06

Table 2. Errors obtained by ADI-DQM for problem 1 with t=0.01 , $\Delta t = 0.001$ and $Re = 150$

$N \times M$	*Errorsof u_1 -computing WCs by*		*Errors of u_2 -computing WCs by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	5.449219E-06	2.464857E-06	8.564094E-06	4.634663E-06
10×10	3.460349E-06	4.545672E-07	7.989946E-06	3.089863E-06
15×15	3.301662E-06	2.039572E-08	2.259864E-05	1.156992E-05

Table 3. Errors obtained by ADI-DQM for problem 1 with t=0.01 , $\Delta t = 0.001$ and $Re = 200$

$N \times M$	*Errors of* u_1*-computing* WCs *by*		*Errors of* u_2 *-computing* WCs *by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	9.440624E-06	6.796386E-06	9.931340E-06	5.435037E-06
10×10	7.790942E-06	1.507044E-06	9.991972E-06	3.693211E-06
5×15	7.583711E-06	8.760431E-07	3.106465E-05	1.276037E-05

Table 4. Errors obtained by DQM for problem 1 with t=0.01 , $\Delta t = 0.001$ and $Re = 100$

$N \times M$	*Errors of* u_1*-computing* WCs *by*		*Errorsof* u_2*-computing* WCs *by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	5.298339E-06	2.743309E-06	1.713128E-05	5.743309E-06
10×10	3.057677E-06	1.174345E-06	1.729607E-05	5.385957E-06
15×15	3.093096E-06	9.361766E-07	1.993608E-05	5.448788E-06

Table 5. Errors obtained by DQM for problem 1 with t=0.01 , $\Delta t = 0.001$ and $Re = 150$

$N \times M$	*Errors of* u_1*-computing* WCs *by*		*Errors of* u_2*-computing* WCs *by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	7.653467E-06	6.786856E-06	1.973118E-05	8.780856E-06
10×10	6.939782E-06	5.213346E-06	2.221854E-05	7.213346E-06
15×15	6.781477E-06	5.463978E-06	2.354824E-05	1.465371E-06

Table 6. Errors obtained by DQM for problem 1 with t=0.01 , $\Delta t = 0.001$ and $Re = 200$

$N \times M$	*Errors of* u_1*-computing* WCs *by*		*Errors of* u_2*-computing* WCs *by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	1.398145E-05	1.277017E-05	3.277547E-05	1.277017E-05
10×10	1.214664E-05	9.448684E-06	3.214664E-05	9.448684E-06
15×15	1.211550E-05	9.334703E-06	4.211550E-05	2.336160E-05

Problem 2. ([2])

We consider Burger equation (1.1) with $\alpha = -2$, $Re = 1$, $L = 1$ and initial conditions in the following form:

$$u_1(x,y,0) = \frac{1}{2} - \frac{x+y}{1+x+y}, \qquad u_2(x,y,0) = \frac{1}{2} + \frac{x+y}{1+x+y} \tag{4.3}$$

The exact solution is given by

$$u_1(x,y,t) = \frac{1}{2} - \frac{x+y+t}{1+x+y+t}, \qquad u_2(x,y,t) = \frac{1}{2} + \frac{x+y+t}{1+x+y+t} \tag{4.4}$$

The boundary conditions can be obtained easily from (4.4) by using $x, y = 0, 1.$ In this problem, we found numerical results for $u_1$and u_2 and using equally spaced grid points. In Table 7 we show the errors obtained in solving problem 2, at t=0.01, $\Delta t = 0.001$ and $x, y \in [0,1]$ for different values of h, with the ADI-DQM by using Fourier series expansion and Lagrange interpolated polynomials to compute the weighted coefficients respectively, and also in Table 8 we show the errors obtained in solving problem 2 with the DQM by using Fourier series expansion and Lagrange interpolated polynomials to compute the weighted coefficients respectively. The numerical results given in tables 7 and 8 are confirm that the ADI-DQM and DQM by using Fourier series expansion to compute the weighting coefficients have a high accuracy, good convergence compare with using Lagrange interpolated polynomials to computing the weighting coefficients.

Table 7. Errors obtained by ADI-DQM for problem 2 with t=0.01 , $\Delta t = 0.0001$

$N \times M$	*Errorsof u_1-computing WCs by*		*Errors of u_2-computing WCs by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	1.308369E-05	1.490516E-06	3.824992E-05	2.855411E-05
10×10	2.287172E-05	5.225470E-06	9.904485E-05	4.198348E-05
15×15	5.397098E-05	1.887698E-05	1.531332E-04	5.444493E-05

Table 8. Errors obtained by DQM for problem 2 with t=0.01 , $\Delta t = 0.0001$

$N \times M$	*Errors of u_1-computing WCs by*		*Errors of u_2-computing WCs by*	
	Lagrange interpolated polynomials	Fourier series expansion	Lagrange interpolated polynomials	Fourier series expansion
5×5	2.361438E-05	1.439153E-05	5.866786E-05	3.980695E-05
10×10	6.637169E-05	1.980695E-05	3.561892E-04	5.254019E-05
15×15	4.092967E-05	2.790255E-05	7.585582E-04	6.472076E-05

The results confirm that ADI-DQM has been , high accuracy, good convergence compare with DQM.

Conclusions

In this work, we employed the ADI-DQM and DQM by using Fourier series expansion and Lagrange interpolated polynomials for computing the weighted coefficients. The methods are applied for solving the Burger equation in two dimensions successfully. The numerical results show that the Fourier series expansion with ADI-DQM and DQM has the higher accuracy and good convergence comparing with using Lagrange interpolated polynomials with ADI-DQM and DQM. The results show that the ADI-DQM and DQM have reasonable accuracy with increases the number of grid points.

References

1. Al-Saif A.S.J. and Al-Kanani M.J.," Numerical study of linear and non-linear convection-diffusion problem by ADI-DQM", J. of Basrah Researches, 37(4)(2011) 26-34

2. Al-Saif A.S.J. and Al-Kanani M.J.," Alternating direction implicit formulation of the differential quadrature method for solving Burger equation", Int. J. Modern Math. Sci. 3(1), 2012, 1-11.

3. Al-Saif A.S.J. and Al-Kanani M.J.," Alternating direction implicit formulation of the differential quadrature method for solving unsteady state two-dimensional convection-diffusion equation", Gen. Math. Notes, 7(1), 2011, 41-50.

4. Al-Kanani M.J,".," Alternating direction implicit formulation of the differential quadrature method for solving convection-diffusion equation ", M. Sc. Thesis, Basrah University, Iraq, 2011.

5. Bellman R, Kashef BG, and Casti J.,"Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations. J. Comput. Phys, 10(1972) 40-52.

6. Basto M., Semiao V., Calheiros F.," Dynamics and synchronization of numerical solutions of the Burgers equation", J. Comput. Appl. Math. 231(2009) 793-803.

7. Peaceman D. and Rachford H., "The numerical solution of elliptic and parabolic

differential equations", Journal of SIAM, 3(1955) 28-41.

8. Quan JR and Chang CT.,"New insights in solving distributed system equations by the quadrature methods-I.", Comput Chem Engrg., 13(1989a)779-788.

9. Quan JR and Chang CT., "New insights in solving distributed system equations by the quadrature methods-II.", Comput Chem Engr., 13(1989b) 1017-1024.

10. Rashidi M. M., Erfani E.," New analytical method for solving Burgers' and nonlinear heat transfer equations and comparison with HAM", Comput. Phys Commun. 180(2009)1539-1544.

11. Shu C.,"Differential quadrature and its application in engineering", Springer-Verlag, London, (2000).

12. Srivastava V. K., Tamsir M., Bhardwaj U. and Sanyasiraju Y. ," Crank-Nicolson Scheme for Numerical Solutions of Two-dimensional Coupled Burgers’ Equations", International Journal Scientific Engineering Research,2(5)(2011)1-7.