Research Article Impact Factor: 0.621 ISSN: 2319-507X
AS Gudadhe, IJPRET, 2014; Volume 2 (9): 532-537 IJPRET

INTERNATIONAL JOURNAL OF PURE AND APPLIED RESEARCH IN ENGINEERING AND TECHNOLOGY

A PATH FOR HORIZING YOUR INNOVATIVE WORK

CONVOLUTION OF EXTENDED FRACTIONAL FOURIER TRANSFORM

A. S. GUDADHE[1], NAVEEN A[2]

1. Govt. Vidarbha Institute of Science and Humanities, Amravati. (M. S.).
2. Vidyabharati Mahavidyalaya, Amravati. (M. S.).

Accepted Date: 27/02/2014 ; Published Date: 01/05/2014

Abstract: The extended fractional Fourier transform, which is the generalization of fractional Fourier transform with two more parameters. It has many applications in several areas, including signal processing and optics. The purpose of this paper is to introduce convolution, product theorem and some properties of convolution for the extended fractional Fourier transform.

Keywords: Extended Fractional Fourier Transform, Convolution And Product Theorem.

Corresponding Author: MR. A. S. GUDADHE

Access Online On:

www.ijpret.com

How to Cite This Article:

AS Gudadhe, IJPRET, 2014; Volume 2 (9): 532-537

PAPER-QR CODE

-

INTRODUCTION

The fractional Fourier transform is the generalization of Fourier transform, which has applications in various fields including Signal processing [6] and Optics [4]. The fractional ordered Fourier transform is introduced by Namias [5] in 1980 given by,

$$F^{\alpha}[f(t)](u) = \frac{e^{\left[i\left(\frac{\pi-2\alpha}{4}\right)\right]}}{\sqrt{2\pi sin\alpha}} \int_{-\infty}^{\infty} e^{-i[\frac{(t^2+u^2)}{2}cot\alpha - tucsc\alpha]} f(t)dt \tag{1.1}$$

Juanwen Hua et. al. [3] suggested the extended fractional Fourier transform with two more parameters as,

$$F_{a,b}^{\alpha}[f(t)](u) = F_{a,b}^{\alpha}(u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt \tag{1.2}$$

when $a = b = \sqrt{\frac{-1}{2\pi}}$ it gives (1.1).

In 1997, Almeida [1] gave a definition for convolution theorem for two functions, $x(t)$ and $y(t)$ as,

$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \leftrightarrow |sec\alpha|\, e^{-i\frac{u^2}{2}tan\alpha} \int_{-\infty}^{\infty} X_{\alpha}(v)\, y[(u-v)sec\alpha]e^{i\frac{v^2}{2}tan\alpha} dv$$

Similarly, the product theorem was given by,

$$x(t)y(t) \leftrightarrow \frac{|csc\alpha|}{\sqrt{2\pi}} e^{i\frac{u^2}{2}cot\alpha} \int_{-\infty}^{\infty} X_{\alpha}(v)\, Y_{\frac{\pi}{2}}[(u-v)csc\alpha]e^{-i\frac{v^2}{2}cot\alpha} dv$$

where, $Y(u)$ is Fourier transform of $y(t)$.

In 1998, Zayed [7] had proved convolution theorem and product theorem as,

$$\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{-i\frac{t^2}{2}cot\alpha} \int_{-\infty}^{\infty} x(\tau)e^{i\frac{\tau^2}{2}cot\alpha} y(t-\tau)e^{i\frac{(t-\tau)^2}{2}cot\alpha} d\tau \leftrightarrow e^{-i\frac{u^2}{2}cot\alpha} X_{\alpha}(u)Y_{\alpha}(u)$$

-

$$x(t)y(t)e^{-i\frac{t^2}{2}cot\alpha} \leftrightarrow \sqrt{\frac{1+icot\alpha}{2\pi}}\, e^{-i\frac{u^2}{2}cot\alpha} \int_{-\infty}^{\infty} X_\alpha(v)e^{-i\frac{v^2}{2}cot\alpha} Y_\alpha(u-v)e^{-i\frac{(u-v)^2}{2}cot\alpha}dv$$

Motivated by the above work, in this paper we have introduced a new convolution for the extended fractional Fourier transform.

2. CONVOLUTION THEOREM:

First we have introduced the convolution for extended fractional Fourier transform as follows.

Definition: For any function $f(t)$, let the functions $\tilde{f}(t)$ and $\tilde{\tilde{f}}(t)$ be $\tilde{f}(t) = f(t)e^{i\pi a^2t^2cot\alpha}$ and $\tilde{\tilde{f}}(t) = f(t)e^{-i\pi b^2t^2cot\alpha}$. Now the convolution of two functions f and g denoted by $f \star g$ is defined as,

$$h(t) = (f \star g)(t) = e^{-i\pi a^2t^2cot\alpha}(\tilde{f} * \tilde{g})(t) \tag{2.1}$$

where $*$ is the usual convolution operation.

Also we define the product operation $\otimes$ by,

$$(f \otimes g)(t) = e^{i\pi b^2t^2cot\alpha}\left(\tilde{\tilde{f}} * \tilde{\tilde{g}}\right)(t) \tag{2.2}$$

THEOREM 2.1: Let $h(t) = (f \star g)(t)$ and $F^\alpha_{a,b}, G^\alpha_{a,b}$ and $H^\alpha_{a,b}$ denote the extended fractional Fourier transform of f, g and h respectively, then

$$H^\alpha_{a,b} = F^\alpha_{a,b}(u)G^\alpha_{a,b}(u)e^{-i\pi b^2u^2cot\alpha} \tag{2.3}$$

Proof: By (1.2) and (2.1)

$$H^\alpha_{a,b}(\mathrm{u}) = \int_{-\infty}^{\infty} h(t)e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]}dt$$

$$= \int_{-\infty}^{\infty} (f \star g)(t)e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]}dt$$

$$= \int_{-\infty}^{\infty} e^{-i\pi a^2t^2cot\alpha}(\tilde{f} * \tilde{g})(t)e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]}dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x)\tilde{g}(t-x)dx.\, e^{i\pi[b^2u^2cot\alpha-2abtucsc\alpha]}dt$$

-

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{i\pi a^2x^2 cot\alpha} g(t-x)e^{i\pi a^2(t-x)^2 cot\alpha}\ e^{i\pi[b^2u^2 cot\alpha - 2abtucsc\alpha]}\,dtdx.$$

Replacing $t - x = v \Rightarrow t = x + v\ and\ dt = dv$

$$H_{a,b}^{\alpha}(\text{u}) = e^{-i\pi b^2u^2 cot\alpha}\int_{-\infty}^{\infty} f(x)\,e^{i\pi(a^2x^2+b^2u^2)cot\alpha - 2i\pi abxucsc\alpha}\,dx.$$

$$\int_{-\infty}^{\infty} g(v)\,e^{i\pi(a^2v^2+b^2u^2)cot\alpha - 2i\pi abvucsc\alpha}\,dv$$

Therefore this completes the proof of

$$H_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}(u)G_{a,b}^{\alpha}(u)e^{-i\pi b^2u^2 cot\alpha}$$

THEOREM 2.2: Let $h(t) = (f \star g)(t)$ and $F_{a,b}^{\alpha}, G_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of f, g and h respectively. Then

$$F_{a,b}^{\alpha}\left[f(t)g(t)e^{i\pi a^2t^2 cot\alpha}\right](u) = \frac{1}{c_\alpha}\left(F_{a,b}^{\alpha} \otimes G_{a,b}^{\alpha}\right)(u) \qquad (2.4)$$

where $c_\alpha = \frac{1}{abcsc\alpha}$

Proof: by (2.2)

$$\left(F_{a,b}^{\alpha} \otimes G_{a,b}^{\alpha}\right)(u) = e^{i\pi b^2u^2 cot\alpha}\left(\tilde{\tilde{F}}_{a,b}^{\alpha} * \tilde{\tilde{G}}_{a,b}^{\alpha}\right)(u)$$

$$= e^{i\pi b^2u^2 cot\alpha}\int_{-\infty}^{\infty}\tilde{\tilde{F}}_{a,b}^{\alpha}(x)\,\tilde{\tilde{G}}_{a,b}^{\alpha}(u-x)dx$$

$$= e^{i\pi b^2u^2 cot\alpha}\int_{-\infty}^{\infty} F_{a,b}^{\alpha}(x)\,e^{-i\pi b^2x^2 cot\alpha}G_{a,b}^{\alpha}(u-x)e^{-i\pi b^2(u-x)^2 cot\alpha}dx$$

$$= e^{i\pi b^2u^2 cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\pi(a^2t^2+b^2x^2)cot\alpha - 2i\pi abtxcsc\alpha}\ e^{-i\pi b^2x^2 cot\alpha}e^{-i\pi b^2(u-x)^2 cot\alpha}$$

$$G_{a,b}^{\alpha}(u-x)dt.dx.$$

Substituting $u - x = v$, then $x = u - v$ and $dx = -dv$

Therefore

-

$(F^{\alpha}_{a,b} \otimes G^{\alpha}_{a,b})(u)$

$$= e^{i\pi b^2u^2 cot\alpha} \int_{-\infty}^{\infty} e^{2i\pi a^2t^2 cot\alpha - 2i\pi abtucsc\alpha} \int_{-\infty}^{\infty} G^{\alpha}_{a,b}(v)\, e^{-i\pi(a^2t^2+b^2v^2)cot\alpha + 2i\pi abtvcsc\alpha} dv f(t) dt$$

By inversion formula [2]

$$= e^{i\pi b^2u^2 cot\alpha} \int_{-\infty}^{\infty} e^{2i\pi a^2t^2 cot\alpha - 2i\pi abtucsc\alpha} \frac{1}{abcsc\alpha} g(t) f(t) dt$$

Therefore

$$(F^{\alpha}_{a,b} \otimes G^{\alpha}_{a,b})(u) = c_{\alpha} F^{\alpha}_{a,b}\left[f(t)g(t)e^{i\pi a^2t^2 cot\alpha}\right](u) \qquad \text{where } c_{\alpha} = \frac{1}{abcsc\alpha}$$

Hence proved.

3. PROPERTIES OF CONVOLUTION OF EXTENDED FRACTIONAL FOURIER TRANSFORM:

In this section we have proved some properties of convolution of extended FrFT. eg. commutative, associative and distributive laws.

3.1 Commutative law: If $F^{\alpha}_{a,b}$ *and* $G^{\alpha}_{a,b}$ denote the extended fractional Fourier transform of f *and* g respectively. Then

$$(f \star g)(t) = (g \star f)(t)$$

Proof: Being very simple proof is omitted.

3.2 Associative law: If $F^{\alpha}_{a,b}$, $G^{\alpha}_{a,b}$ and $H^{\alpha}_{a,b}$denote the extended fractional Fourier transform of f, g *and* h respectively. Then

$$[(f \star g) \star h](t) = [f \star (g \star h)](t)$$

Proof: Proof is simple hence omitted.

3.3 Distributive law: If $F^{\alpha}_{a,b}$, $G^{\alpha}_{a,b}$ and $H^{\alpha}_{a,b}$denote the extended fractional Fourier transform of f, g *and* h respectively. Then

$$[f \star (g + h)](t) = [(f \star g) + (f \star h)](t)$$

Proof: Proof is simple hence omitted.

-

CONCLUSION: In this paper we have defined a new convolution for the extended fractional Fourier transform and proved the theorem for convolution of extended FrFT. Then we have also proved product theorem and some properties for convolution of extended FrFT, which are useful in filter designing.

REFERENCES:

1. Almeida L. B. (1997): Product and convolution theorems for the fractional Fourier transform, IEEE Signal Processing letters, Vol. 4, No. 1.

2. Gudadhe A. S. and Naveen A.(2013): Extended fractional Fourier transform of distributions of compact support. Journal of Science and Arts, Year 13, No. 4, pp. 345-354.

3. Hua J., Liren Liu and Guoqiang Li (Dec-1997): Extended fractional Fourier transforms, J. Opt. Soc. Am. A/Vol. 14, No. 12.

4. Mendlovic D. and Ozaktas H. M. (1993): Fractional Fourier transform and their optical implementation. I. J. Opt. Soc. Am. A 10,1875-1881.

5. Namias V. (1980): The fractional order Fourier transform and its application to quantum mechanics, J. Inst. Math. Appl. 25, 241-265.

6. Ozaktas H. M., Zulevsky Z. and Kutay M. A. (2001): The fractional Fourier transform with Applications in Optics and Signal Process, John Wiley and Sons Chichester.

7. Zayed A. I. (1998): A Convolution and Product Theorem for the fractional Fourier transform, IEEE Signal Processing letters, Vol. 5, No. 4.