INTERNATIONAL JOURNAL OF PURE AND APPLIED RESEARCH IN ENGINEERING AND TECHNOLOGY

A PATH FOR HORIZING YOUR INNOVATIVE WORK

REPRESENTATION THEOREM FOR THE DISTRIBUTIONAL FOURIER-FINITE MELLIN TRANSFORM

V. D. SHARMA[1], A. N. RANGARI[2]

1. Department of Mathematics, Arts, Commerce and Science College, Amravati- 444606(M.S), India.
2. Department of Mathematics, Adarsh College, Dhamangaon Rly. - 444709 (M.S), India.

Accepted Date: 05/03/2015; Published Date: 01/05/2015

Abstract: Integral transformations have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics, and engineering science. The origin of the integral transform is the Fourier and Mellin transform. Integral transformation is one of the well-known techniques used for function transformation. The transforms we will be studying in this part of the course are mostly useful to solve differential and to a lesser extent, integral equations. The specific Fourier and Finite Mellin transforms being used may differ from application to application. Fourier-Finite mellin Transform has found numerous applications in various fields. We tried to develop a new type of transform Fourier-Finite Mellin transform on the same lines. We are trying to use this newly developed transform to solve various problems. In this paper Fourier-Finite Mellin Transform is generalized in the distributional sense. Testing function spaces using Gelf and Shilov technique are already developed in our previous papers. The main aim of this paper was to prove Representation Theorem for the distributional Fourier-Finite Mellin Transform.

Keywords: Fourier Transform, Finite Mellin Transform, Fourier- Finite Mellin Transform, Generalized function, Testing function space.

Corresponding Author: MR. V. D. SHARMA

Access Online On:

www.ijpret.com

How to Cite This Article:

V. D. Sharma, IJPRET, 2015; Volume 3 (9): 126-134

PAPER-QR CODE

INTRODUCTION

Mathematics is everywhere in every phenomenon, technology, observation, experiment etc. All we need to do is to understand the logic hidden behind [1]. Communication is all based on Mathematics, be it digital, wired or wireless using Fourier Transform analysis. Fourier Transform lies at the heart of signal processing and image processing. The Fourier Transform is a tool for solving physical problems. It is applied to optics, crystallography, solving science problems. Fourier Transform can be strongly used in acoustics. It is used to understand how different musical instruments create their different sounds. The Fourier Transform occurs naturally all throughout physics. It is also used in determining organic structures from IR/NMR, Mathematical Statistics [2]. The Fourier Transform is among the most widely used tools for transforming data sequences and functions. Emerging uses of Fourier Transform is in Infrared FTIR Spectroscopy. The fundamental importance of this transform is in quantum mechanics.

The Mellin Transform is used for solving certain classes of singular integral equations [3]. It is used in pattern recognition. The Mellin transformation is a basic tool for analyzing the behavior of many important functions in Mathematics and mathematical physics. Mellin transformation is also used in Statistics. It is also used in time-frequency analysis [4, 5]. In the late 1970's Casasent and Psaltis [1976, 1997] contributed substantially to the implementation of a digital form of the Fourier-Mellin Transform in application using physical lenses. The Fourier-Mellin Transform was implemented on a digital computer and applied towards the recognition and differentiation of images of plant leaves regardless of translation, rotation or scale. Translated, rotated and scaled leaf images from seven species of plant were compared. It also used for movement detection [6].

Many authors studied double transform. Motivated by this we have also defined a new transform namely Fouriet-Finite Mellin Transform and this newely Fouriet-Finite Mellin Transform may be used for image recognition, reconstruction and processing, movement detection and derivation of densities for algebraic combinations of random variables and many more [6].

The Fourier Transform is defined separately as follows:-

The Fourier Transform with parameter s of $f(t)$ denoted by $F[f(t)] = F(s)$ and is given by

$$F[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)dt, \qquad (1.1)$$

For parameter $s > 0$.

The Finite Mellin Transform is separately defined as given bellow:-

The Finite Mellin Transform with parameter p of $f(x)$ denoted by $M_f\left[f(x)\right]=F(p)$ and is given by

$$M_f\left[f(x)\right]=F(p)=\int_0^a\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)f(x)dx$$

, for parameter $p>0$. (1.2)

We combined these two separate transforms and defined a new Fourier-Finite Mellin Transform as follows:-

$$FM_f\left\{f(t,x)\right\}=F(s,p)=\int_{-\infty}^{\infty}\int_0^a e^{-ist}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)f(t,x)dtdx \qquad (1.3)$$

In this paper Fourier-Finite Mellin Transform is extended in the distributional sense. Representation theorem for Fourier-Finite Mellin Transform is also presented.

This paper is organized as follows:-

In Section 2, Definition of Distributional Generalized Fourier-Finite Mellin Transform is given. Testing function spaces are given in section 3. In section 4, we have proved Representation theorem. Lastly the conclusions are given in section 5.

Notations and terminologies are as per Zemanian [7],[8].

2. DISTRIBUTIONAL GENERALIZED FOURIER-FINITE MELLIN TRANSFORMS (FM_fT)

For $f(t,x)\in FM^{*\beta}_{f,b,c,\alpha}$, where $FM^{*\beta}_{f,b,c,\alpha}$ is the dual space of $FM^{\beta}_{f,b,c,\alpha}$. It contains all distributions of compact support. The distributional Fourier-Finite Mellin transform is a function of $f(t,x)$ and is defined as

$$FM_f\left\{f(t,x)\right\}=F(s,p)=\left\langle f(t,x),e^{-ist}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle \qquad (2.1)$$

where, for each fixed t $(0<t<\infty)$, x $(0<x<\infty)$, $s>0$ and $p>0$, the right hand side of (2.1) has a sense as an application of $f(t,x)\in FM^{*\beta}_{f,b,c,\alpha}$ to $e^{-ist}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\in FM^{\beta}_{f,b,c,\alpha}$.

3. TESTING FUNCTION SPACES

3.1. The Space $FM^{\beta}_{f,b,c,\alpha}$

This space is given by

$$FM^{\beta}_{f,b,c,\alpha} = \left\{\phi : \phi \in E_+ / \rho_{b,c,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \phi(t,x) \right| \le CA^k k^{k\alpha} B^l l^{l\beta} \right\}$$

(3.1)

where, $k,l,q = 0,1,2,3,..........$, and the constants A,B depends on the testing function ϕ .

3.2. The Space $FM_{f,b,c,\gamma}$

It is given by

$$FM_{f,b,c,\gamma} = \left\{\phi : \phi \in E_+ / \xi_{b,c,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \phi(t,x) \right| \le C_{lk} A^q q^{q\gamma} \right\}$$

(3.2)

where, $k,l,q = 0,1,2,3,..........$, and the constants depend on the testing function ϕ

where, $\lambda_{b,c}(x) = \{x^{+b}, 0<x<1, x^{+c}, 1<x<a$

4. REPRESENTATION THEOREM

Let $f(t,x)$ be an arbitrary element of $FM^{*\beta}_{f,b,c,\alpha}$ and $\phi(t,x)$ be an element of $D(I)$, the space of infinitely differentiable function with compact support on I . Then there exists a bounded measurable functions $g_{m,n}(t,x)$ defined over I such that

$$\langle f,\phi\rangle = \left\langle \sum_{m=0}^{r+1}\sum_{n=0}^{\nu+1} (-1)^{m+n} t^k x^{b+q+1} \frac{\partial^{m+n}}{\partial t^m \partial x^n} g_{m,n}(t,x), \phi(t,x) \right\rangle$$

where k is a fixed real number and r and ν are appropriate non-negative integers satisfying $m \le r+1$ and $n \le \nu+1$.

Proof:- Let $\{\gamma_{b,c,k,l,q}\}_{k,l,q=0}^{\infty}$ be the sequence of seminorms. Let $f(t,x)$ and $\phi(t,x)$ be arbitrary elements of $FM^{*\beta}_{f,b,c,\alpha}$ and $D(I)$ respectively. Then by boundedness property of generalized function by Zemanian [7], pp.52, we have for an appropriate constant C and a non-negative integer r and ν satisfying $|l| \le r$ and $|q| \le \nu$

$$|\langle f,\phi\rangle| \le C \max_{\substack{|l|\le r\\ |q|\le \nu}} \gamma_{b,c,k,l,q}\phi(t,x) \le C \max_{\substack{|l|\le r\\ |q|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<a}} t^k x^{b+q+1} \left|D_t^l D_x^q \phi(t,x)\right|$$

$$\le C \max_{\substack{|l|\le r\\ |q|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<a}} t^k x^{b+q+1} \left|\sum_{m=0}^{l}\sum_{n=0}^{q} B_n \frac{\partial^{m+n}}{\partial t^m \partial x^n}\phi(t,x)\right|$$

$$\le C' \max_{\substack{|l|\le r\\ |q|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<a}} t^k x^{b+q+1} \max_{\substack{m\le l\\ n\le q}} \frac{\partial^{m+n}}{\partial t^m \partial x^n}\phi(t,x)$$

where C' is a constant which depends only on m,n and hence l,q, so

$$|\langle f,\phi\rangle| \le C'' \max_{\substack{|m|\le r\\ |n|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<a}} t^k x^{b+q+1} \frac{\partial^{m+n}}{\partial t^m \partial x^n}\phi(t,x) \tag{4.1}$$

Now let us set

$$\phi_{r,\nu}(t,x) = t^k x^{b+q+1}\phi(t,x), \quad m \le r, n \le \nu \tag{4.2}$$

Then clearly $\phi_{r,\nu}(t,x) \in D(I)$.

Also $\phi(t,x) = t^{-k} x^{-b-q-1}\phi_{r,\nu}(t,x)$ (4.3)

On differentiating (4.3) partially with respect to t and x successively we get,

$$\frac{\partial^2 \phi}{\partial t \partial x} = t^{-k} x^{-b-q-1}\left[\frac{k(q+1+b)}{tx}\phi_{r,\nu} - \frac{(q+1+b)}{x}\frac{\partial \phi_{r,\nu}}{\partial t} - \frac{k}{t}\frac{\partial \phi_{r,\nu}}{\partial x} + \frac{\partial^2 \phi_{r,\nu}}{\partial t \partial x}\right]$$

Let us suppose that in I, $Sup\phi = Sup\phi_{r,\nu} = [A,B]$.

Then since $t^{-k}x^{-b-q-1} > 0$

$$\left|\frac{\partial^2\phi}{\partial t\partial x}\right| \le t^{-k}x^{-b-q-1}\left\{\frac{|k(q+1+b)|}{AB}|\phi_{r,\nu}| + \frac{|q+1+b|}{B}\left|\frac{\partial\phi_{r,\nu}}{\partial t}\right| + \frac{|k|}{A}\left|\frac{\partial\phi_{r,\nu}}{\partial x}\right| + \left|\frac{\partial^2\phi_{r,\nu}}{\partial t\partial x}\right|\right\}$$

$$\le C'''t^{-k}x^{-b-q-1}\left\{|\phi_{r,\nu}| + \left|\frac{\partial\phi_{r,\nu}}{\partial t}\right| + \left|\frac{\partial\phi_{r,\nu}}{\partial x}\right| + \left|\frac{\partial^2\phi_{r,\nu}}{\partial t\partial x}\right|\right\}$$

Where, $C''' = \max\left[\frac{|k(q+1+b)|}{AB}, \frac{|q+1+b|}{B}, \frac{|k|}{A}, 1\right]$

If C^{iv} is a constant which depends on b, k and $q+1$ then

$$\left|\frac{\partial^2\phi}{\partial t\partial x}\right| \le C^{iv}t^{-k}x^{-b-q-1}\left|\frac{\partial^2\phi_{r,\nu}(t,x)}{\partial t\partial x}\right|$$

Hence by induction we prove that in I , for obvious constant C^{v} .

$$\left|\frac{\partial^{m+n}\phi}{\partial t^m\partial x^n}\right| \le C^{v}t^{-k}x^{-b-q-1}\sum_{\substack{c\le m\\ d\le n}}\left|\frac{\partial^{c+d}}{\partial t^c\partial x^d}\phi_{r,\nu}(t,x)\right|$$

Substituting this into (4.1)

$$|\langle f,\phi\rangle| \le C^{vi} \max_{\substack{m\le r\\ n\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<a}}\left|\frac{\partial^{c+d}}{\partial t^c\partial x^d}\phi_{r,\nu}(t,x)\right|$$ where, $c\le m$ and $d\le n$ (4.4)

Now we can write

$$\sup_{\substack{0<t<\infty\\ 0<x<a}}|\phi(t,x)| \le \sup_{\substack{0<t<\infty\\ 0<x<a}}\left|\int_t^\infty\int_x^a \frac{\partial^2}{\partial t\partial x}\phi(t,x)dtdx\right|$$

$$\le \left\|\frac{\partial^2}{\partial t\partial x}\phi(t,x)\right\|_{L'XL'}$$

(4.5) Hence from (4.4)

$$|\langle f,\phi\rangle| \le C^{vi} \max_{\substack{m\le r+1\\ n\le \nu+1}} \sup_{\substack{0<t<\infty\\ 0<x<a}}\left\|\frac{\partial^{m+n}}{\partial t^m\partial x^n}\phi_{r,\nu}(t,x)\right\|_{L'XL'}$$

Let the product space $L' \times L'$ be denoted by $(L')^2$. We consider the linear one-to-one mapping $\tau : \phi \to \left\{ \frac{\partial^{m+n}}{\partial t^m \partial x^n} \phi \right\}_{\substack{m \le r+1 \\ n \le \nu+1}}$ of $D(I)$ into $(L')^2$. In view of (4.5) we see that the linear functional $\tau : \phi_{r,\nu} \to \langle f, \phi \rangle$ is continuous on $\tau D(I)$ for the topology induced by (L'). Hence by Hahn-Banach theorem, it can be a continuous linear functional in the whole of $(L')^2$. But the dual of $(L')^2$ is isomorphic with $(L^\infty)^2$ [9] pp.214 and 259, therefore there exist two L^∞ functions $g_{m,n}(m \le r+1, n \le \nu+1)$ such that,

$$\langle f, \phi \rangle = \sum_{\substack{m \le r+1 \\ n \le \nu+1}} \left\langle g_{m,n}, \frac{\partial^{m+n}}{\partial t^m \partial x^n} \phi_{r,\nu}(t,x) \right\rangle$$

By (4.2), we have

$$\langle f, \phi \rangle = \sum_{\substack{m \le r+1 \\ n \le \nu+1}} \left\langle g_{m,n}, \frac{\partial^{m+n}}{\partial t^m \partial x^n} t^k x^{b+q+1} \phi(t,x) \right\rangle$$

Now by using property of differentiation of a distribution and property of multiplication of a distribution by an infinitely smooth function,

$$\langle f, \phi \rangle = \sum_{\substack{m \le r+1 \\ n \le \nu+1}} \left\langle (-1)^{m+n} t^k x^{b+q+1} \frac{\partial^{m+n}}{\partial t^m \partial x^n} g_{m,n}(t,x), \phi(t,x) \right\rangle$$

where $g_{m,n}(t,x)$ are bounded measurable functions defined over $I = (0, \infty)$. Therefore

$$f(t,x) = \sum_{\substack{m \le r+1 \\ n \le \nu+1}} (-1)^{m+n} t^k x^{b+q+1} \frac{\partial^{m+n}}{\partial t^m \partial x^n} g_{m,n}(t,x)$$

5. CONCLUSION

Since Fourier and Finite Mellin transforms has found numerous applications in various fields. We tried to develop a new type of transform, Fourier-Finite Mellin transform on the same lines. In this paper Fourier-Finite Mellin transform is generalized in the distributional sense. Testing function spaces using Gelf and Shilov technique are already developed in our previous papers. The main aim of this paper was to prove Representation Theorem for the distributional Fourier-Finite Mellin transform and we proved it.

REFERENCES

1. Gupta Anupama, Fourier Transform and Its Application in Cell Phones, International Journal of Scientific and Research Publications, Volume 3, Issue 1, pp. 1-2 January 2013.

2. Debnath Lokenath and Bhatta Dambaru, Integral Transforms and their Applications, Chapman and Hall/CRC Taylor and Francis Group Boca Raton London, New York, 2007.

3. Alireza Ansari, Generalized Product Theorem for the Mellin Transform And its Applications, Kragujevac Journal of Mathematics, Vol. 36 No. 2 pp.287-298, 2012.

4. S. M. Khairnar, R. M. Pise and J. N. Salunke, Generalized finite mellin integral transforms, Archives of Applied Science Research, 4(2):1117-1134, 2012.

5. S. M. Khairnar, R. M. Pise, and J. N. Salunke, "Applications of the Mellin Type integral Transform in the range $(\frac{1}{a},\infty)$," International Journal of Mathematical Sciences and Applications, Vol. 1 No. 1,pp.1-17 (January, 2011).

6. John Graham le Maistre Pratt, "Application of the Fourier-Mellin transform to translation-, rotation- and scale-invariant plant leaf identification," Thesis, McGill University, Montreal, July 2000.

7. Zemanian A. H. Generalized integral transform, Inter science publisher, New York, 1968.

8. Zemanian A.H., Distribution theory and transform analysis, McGraw Hill, New York, 1965.

9. Treves, f, Topological vector space, distribution and kernels, Academic press, New York, (1967).

10. Sharma, V. D. and Rangari A. N., Fourier-Finite Mellin Transform and Its Analytical Study, International J. of Applied Computational Science & Mathematics, Vol.4, No. 1 (2014) pp. 25-30 .

11. Sharma, V. D. and Rangari A. N., Analyticity of Generalized Fourier-Finite Mellin Transform, International J. of Mathematical Education, Vol.4, No. 1 (2014) pp. 7-12.

12. Sharma, V. D. and Rangari A. N., Operational Calculus on Fourier-Finite Mellin Transform, International J. of Engineering and Innovative Technology, Vol.3, Issue 4. Oct. 2013 pp. 161-164 .

13. Gulhane, P. A. and Gudadhe, A. S., Representation Theorem for the Distributional Laplace-Stieltjes Transform, Science Journal of GVISH, Vol. II, 2005, 29-32.

14. Pathak, R. S., A Representation Theorem for a class of stieltjes Transformable Generalized functions Indian Mathematical Society, (1974), pp392-342.

15. Pathak, R. S., A Course in Distribution Theory and Applications, CRC Press 2001.