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A PATH FOR HORIZING YOUR INNOVATIVE WORK

SOLUTION OF NON-HOMOGENEOUS DIFFERENTIAL EQUATION WITH THREE AND FOUR FRACTIONAL DERIVATIVES

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Abstract: - The linear non-homogeneous ordinary differential equations with three (four) order fractional derivative are considered. Using the direct and inverse Laplace transforms the series solutions of these equations are investigated. Several examples are then given to demonstrate the validity of our main results.

Keywords: Fractional differential equations, Riemann–Liouville derivative, Caputo derivative, Laplace transforms.

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1. INTRODUCTION

Fractional differential equations are generalization of the ordinary differential equation to arbitrary non-integer order. Fractional differential equations arise in many complex systems in nature and society with many dynamics, such as rheology, porous media, viscoelasticity, electrochemistry, electromagnetism, signal processing, dynamics of earthquakes, optics, geology, viscoelastic materials, biosciences, bioengineering, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, fluid mechanics, electromagnetic waves, nonlinear control, control of power electronic, converters, chaotic dynamics, polymer science, proteins, polymer physics, electrochemistry, statistical physics, thermodynamics, neural networks, and many more [9,11,12,18,19,25,28] . In recent years, there has been a significant development in the techniques of solving fractional differential equations, some recent contributions can be seen in [1-11,13-17,20-28], and the references therein. A wide description of the problem of the existence and uniqueness of solutions of Cauchy-type problems for fractional order differential equations together with its applications can be found in the literature [1, 6, 7, 23, 26, 21-28].

2. Definitions and Preliminaries

1. The fractional derivative of a causal function $f\ (t)$ ([1,2]) is define by

$$\frac{d^{\alpha}}{d\,t^{\alpha}}\,f(t)=\begin{cases} f^{(n)}(t) & if \quad \alpha = n \in N, \\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}\,d\,t & if \quad n-1<\alpha<n, \end{cases}$$

where the Euler gamma function $\Gamma(.)$ is defined by

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\,d\,t \qquad (x>0).$$

2. The Laplace transform of a function $f\ (t),\ t\ \in (0,\infty)$ is defined by

$$L\big[f\,(t)\big](s)=F\,(s)=\int_0^{\infty} e^{-st}\ f\,(t)\,d\,t \qquad (s\in C).$$

3. The Riemann – Liouville fractional derivatives $D_{a+}^{\alpha}\,y$ and $D_{b-}^{\alpha}\,y$ of order $\alpha\in C\ (\Re(\alpha)\geq 0)$ are defined by

$$\left(D_{a+}^{\alpha} y\right)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d x}\right)^{n} \int_{a}^{x} \frac{y(t) d t}{(x-t)^{\alpha-n+1}} \quad \left(n=[\Re(\alpha)]+1;\ x>a\right)$$

and

$$\left(D_{b-}^{\alpha} y\right)(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{d x}\right)^{n} \int_{x}^{b} \frac{y(t) d t}{(t-x)^{\alpha-n+1}} \quad \left(n=[\Re(\alpha)]+1;\ x<b\right)$$

Respectively, where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$.

4. The pochhammer symbol (or the shifted factorial, since $(1)_n = n!$ for $n \in \mathrm{N}_0 = \{0,1,2,\ldots\}$) given by

$$(\lambda)_n = \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in N_0 / \{0\}). \end{cases}$$

5. The binomial coefficients are defined by

$$\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda-n)!} = \frac{\lambda(\lambda-1)(\lambda-n+1)}{n!}$$

Where λ and n are integers. Observe that $0! = 1$, then

$$\binom{\lambda}{0} = 1, \quad \binom{\lambda}{\lambda} = 1 \quad \text{and} \quad (1-z)^{-\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} z^r = \sum_{r=0}^{\infty} \binom{\lambda+r-1}{r} z^r.$$

6. $L\left[D^{\alpha} f(t)\right](s)=s^{\alpha} L\left[D^{\alpha} f(t)\right](s)-\sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0)$,

Where $\alpha>0, n-1<\alpha \leq n (n \in N)$, $f(t) \in C^{n}(0, \infty), f^{(n)}(t) \in L_{1}(0, b)$ for any $b>0$ [15].

3. Mean Result

Theorem 3.1: Let $f(t)$ be analytic in $[0,t_1]$, , $n-1<\alpha<n$ and $a,b\in R$

$y^{\alpha+2}(t)+a\,y^{\alpha+1}(t)+b\,y^{\alpha}(t) = f(t)$,

Then

$$y(t)=\sum_{k=1}^{n} y^{(k-1)}(0)\,\frac{t^{k-1}}{\Gamma(k)}+y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k\,t^{2k+n}}{k!}\frac{\Gamma(k+r+1)(-at)^r}{\Gamma(n+r+2k+1)r!}$$
$$+y^{(n+1)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k\,t^{2k+n+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)\,(-at)^r}{\Gamma(n+r+2k+2)r!}$$
$$+a\,y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k\,t^{n+2k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)\,(-at)^r}{\Gamma(n+r+2k+2)r!}$$
$$+\sum_{l=1}^{\infty}a_l\,l!\sum_{k=0}^{\infty}\frac{(-b)^k\,t^{2k+\alpha+l+2}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at)^r}{\Gamma(r+2k+\alpha+l+3)r!}$$

Proof: Since $f(t)$ is analytic in $[0,t_1]$, one can expand it as $f(t)=\sum_{i=0}^{\infty}a_i\,t^i$ where $a_i=\frac{f^{(i)}(0)}{i!},\quad \forall i=0,1,2,\ldots$

Now, applying the Laplace transform we have

$$s^{\alpha+2}\,\bar{y}(s)-\sum_{k=1}^{n+2}s^{\alpha+2-k}\,y^{(k-1)}(0)+a\,s^{\alpha+1}\,\bar{y}(s)-a\sum_{k=1}^{n+1}s^{\alpha+1-k}\,y^{(k-1)}(0)+b\,s^{\alpha}\,\bar{y}(s)$$
$$-b\sum_{k=1}^{n}s^{\alpha-k}\,y^{(k-1)}(0)=\sum_{l=1}^{\infty}a_l\,\frac{l!}{s^{l+1}}$$

$$\bar{y}(s)=\sum_{k=1}^{n}s^{-k}\,y^{(k-1)}(0)+\frac{s^{1-n}}{s^2+as+b}\,y^{(n)}(0)+\frac{s^{-n}}{s^2+as+b}\,y^{(n+1)}(0)$$
$$+a\frac{s^{-n}}{s^2+as+b}\,y^{(n)}(0)+\sum_{l=1}^{\infty}\frac{a_l\,l!\,s^{-\alpha-l-1}}{\left(s^2+as+b\right)}$$

Since

$$\frac{1}{s^2+as+b}=\frac{s^{-1}}{s+a+bs^{-1}}=\frac{s^{-1}}{(s+a)\left(1+\frac{bs^{-1}}{s+a}\right)}$$

$$=\frac{s^{-1}}{(s+a)}\sum_{k=0}^{\infty}\left(\frac{-bs^{-1}}{s+a}\right)^k=\sum_{k=0}^{\infty}\frac{(-b)^k s^{-k-1}}{(s+a)^{k+1}}=\sum_{k=0}^{\infty}\frac{(-b)^k s^{-2k-2}}{(1+as^{-1})^{k+1}}$$

$$=\sum_{k=0}^{\infty}(-b)^k s^{-2k-2}\sum_{r=0}^{\infty}(-as^{-1})^r\binom{k+r}{r}$$

$$=\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r s^{-r-2k-2}$$

By taken inverse Laplace transform, one can have

$$y(t)=\sum_{k=1}^{n}y^{(k-1)}(0)\frac{t^{k-1}}{\Gamma(k)}+y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k}{k!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma(n+r+2k+1)}\frac{t^{n+r+2k}}{r!}$$
$$+y^{(n+1)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k}{k!}\sum_{r=0}^{\infty}\frac{(k+r)!\,(-a)^r}{\Gamma(n+r+2k+2)}\frac{t^{n+r+2k+1}}{r!}$$
$$+a\,y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k}{k!}\sum_{r=0}^{\infty}\frac{(k+r)!\,(-a)^r}{\Gamma(n+r+2k+2)}\frac{t^{n+r+2k+1}}{r!}$$
$$+\sum_{l=1}^{\infty}a_l\,l!\sum_{k=0}^{\infty}\frac{(-b)^k}{k!}\sum_{r=0}^{\infty}\frac{(k+r)!\,(-a)^r}{\Gamma(r+2k+\alpha+l+3)}\frac{t^{r+2k+\alpha+l+2}}{r!}$$

$$y(t)=\sum_{k=1}^{n}y^{(k-1)}(0)\frac{t^{k-1}}{\Gamma(k)}+y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k+n}}{k!}\frac{\Gamma(k+r+1)(-at)^r}{\Gamma(n+r+2k+1)r!}$$
$$+y^{(n+1)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k+n+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)\,(-at)^r}{\Gamma(n+r+2k+2)r!}$$
$$+a\,y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-b)^k t^{n+2k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)\,(-at)^r}{\Gamma(n+r+2k+2)r!}$$
$$+\sum_{l=1}^{\infty}a_l\,l!\sum_{k=0}^{\infty}\frac{(-b)^k t^{2k+\alpha+l+2}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at)^r}{\Gamma(r+2k+\alpha+l+3)r!}$$

Example 3.2: Let $\alpha=\frac{3}{2}, a=\sqrt{3}$ and $b=-8$ in theorem (3.1), then the solution of $y^{\frac{7}{2}}(t)+\sqrt{3}\, y^{\frac{5}{2}}(t)-8\, y^{\frac{3}{2}}(t) = e^{\sqrt{3}\,t}$ is

$$y(t)=\sum_{k=1}^{n} y^{(k-1)}(0)\,\frac{t^{k-1}}{\Gamma(k)}+\, y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-8)^k\, t^{2k+n}}{k!}\frac{\Gamma(k+r+1)(-\sqrt{3}\,t)^r}{\Gamma(n+r+2k+1)r!}$$

$$+\,y^{(n+1)}(0)\sum_{k=0}^{\infty}\frac{(-8)^k\, t^{2k+n+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)\,(-\sqrt{3}\,t)^r}{\Gamma(n+r+2k+2)\,r!}$$

$$+\sqrt{3}\,y^{(n)}(0)\sum_{k=0}^{\infty}\frac{(-8)^k\, t^{n+2k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)\,(-\sqrt{3}\,t)^r}{\Gamma(n+r+2k+2)\,r!}$$

$$+\sum_{l=1}^{\infty}\left(\sqrt{3}\right)_l\, l!\sum_{k=0}^{\infty}\frac{(-8)^k\, t^{2k+l+\frac{7}{2}}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-\sqrt{3}\,t)^r}{\Gamma(r+2k+l+\frac{9}{2})\,r!}$$

Theorem 3.3: Let $f(t)$ be analytic in $[0,t_1]$, $n-1<\alpha<n$ and $a,b\in R$

$$y^{\alpha+3}(t)+a\,y^{\alpha+2}(t)+b\,y^{\alpha+1}(t)+c\,y^{\alpha}(t) = f(t)$$

Then

$$y(t)=\sum_{k=1}^{n} y^{(k-1)}(0)\frac{t^{k-1}}{\Gamma(k)}+\, y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-2}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c\,t^{-1}\right)^J}{J!\,\Gamma(k-J+1)}\;\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a\,t^{-1})^r}{\Gamma[-2k-r-J-n-1]\,r!}$$

$$+\, y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-3}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c\,t^{-1}\right)^J}{J!\,\Gamma(k-J+1)}\;\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a\,t^{-1})^r}{\Gamma[-2k-r-J-n-2]\,r!}$$

$$+\, y^{(n+2)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c\,t^{-1}\right)^J}{J!\,\Gamma(k-J+1)}\;\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a\,t^{-1})^r}{\Gamma[-2k-r-J-n-3]\,r!}$$

$$+a\; y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-3}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c\,t^{-1}\right)^J}{J!\,\Gamma(k-J+1)}\;\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a\,t^{-1})^r}{\Gamma[-2k-r-J-n-2]\,r!}$$

$$+a\ y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k\ t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c t^{-1}\right)^J}{J!\Gamma(k-J+1)}\ \sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a t^{-1})^r}{\Gamma\left[-2k-r-J-n-3\right]r!}$$

$$+b\ y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\ t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c t^{-1}\right)^J}{J!\Gamma(k-J+1)}\ \sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a t^{-1})^r}{\Gamma\left[-2k-r-J-n-3\right]r!}$$

$$+\sum_{l=1}^{\infty}a_l\,l!\sum_{k=0}^{\infty}(-1)^k\,t^{-2k-\alpha-l-5}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(c t^{-1}\right)^J}{J!\Gamma(k-J+1)}\ \sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-a t^{-1})^r}{\Gamma\left[-2k-r-J-\alpha-l-4\right]r!}$$

Proof: Since $f(t)$ is analytic in $[0,t_1]$, one can expand it as $f(t)=\sum_{i=0}^{\infty}a_i t^i$ where $a_i=\frac{f^{(i)}(0)}{i!},\quad \forall i=0,1,2,\ldots$

Now, applying the Laplace transform we have

$$s^{\alpha+3}\,\bar{y}(s)-\sum_{k=1}^{n+3}s^{\alpha+3-k}\ y^{(k-1)}(0)+a\,s^{\alpha+2}\ \bar{y}(s)-a\sum_{k=1}^{n+2}s^{\alpha+2-k}\ y^{(k-1)}(0)+b\,s^{\alpha+1}\,\bar{y}(s)$$
$$-b\sum_{k=1}^{n+1}s^{\alpha+1-k}\ y^{(k-1)}(0)+c\,s^{\alpha}\ \bar{y}(s)-c\sum_{k=1}^{n}s^{\alpha-k}\ y^{(k-1)}(0)=\sum_{l=1}^{\infty}a_l\ \frac{l!}{s^{l+1}}$$

$$\bar{y}(s)=\sum_{k=1}^{n}s^{-k}\ y^{(k-1)}(0)+\frac{s^{2-n}}{s^3+a\,s^2+b\,s+c}\,y^{(n)}(0)+\frac{s^{1-n}}{s^3+a\,s^2+b\,s+c}\ y^{(n+1)}(0)$$
$$+\ \frac{s^{-n}}{s^3+a\,s^2+b\,s+c}\ y^{(n+2)}(0)+a\,\frac{s^{1-n}}{s^3+a\,s^2+b\,s+c}\ y^{(n)}(0)+a\,\frac{s^{-n}}{s^3+a\,s^2+b\,s+c}\ y^{(n+1)}(0)$$
$$+b\,\frac{s^{-n}}{s^3+a\,s^2+b\,s+c}\ y^{(n)}(0)+\sum_{l=1}^{\infty}\frac{a_l\,l!\,s^{-\alpha-l-1}}{\left(s^3+a\,s^2+b\,s+c\right)}$$

Since

$$\frac{1}{s^3+as^2+bs+c}=\frac{s^{-2}}{s+a+bs^{-1}+cs^{-2}}=\frac{s^{-2}}{(s+a)\left(1+\frac{bs^{-1}}{s+a}+\frac{cs^{-2}}{s+a}\right)}$$

$$=\frac{s^{-2}}{(s+a)}\sum_{k=0}^{\infty}\left(\frac{-bs^{-1}-cs^{-2}}{s+a}\right)^k=\frac{s^{-2}}{(s+a)^{k+1}}\sum_{k=0}^{\infty}(-bs^{-1}-cs^{-2})^k$$

$$=\sum_{k=0}^{\infty}(-1)^k\sum_{J=0}^{\infty}(b)^{k-J}c^J\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\ s^{-3-2k-r-J}$$

By taken inverse Laplace transform, one can have

$$
\begin{aligned}
y(t) = &\sum_{k=1}^{n} y^{(k-1)}(0)\frac{t^{k-1}}{\Gamma(k)} + y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{J!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-1-2k-r-J-n]}\frac{t^{-2k-r-J-n-2}}{r!} \\
&+ y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{j!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-2-2k-r-J-n]}\frac{t^{-2k-r-J-n-3}}{r!} \\
&+ y^{(n+2)}(0)\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{j!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-3-2k-r-J-n]}\frac{t^{-2k-r-J-n-4}}{r!} \\
&+a\, y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{J!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-2-2k-r-J-n]}\frac{t^{-2k-r-J-n-3}}{r!} \\
&+a\, y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{J!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-3-2k-r-J-n]}\frac{t^{-2k-r-J-n-4}}{r!} \\
&+b\, y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{J!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-3-2k-r-J-n]}\frac{t^{-2k-r-J-n-4}}{r!} \\
&+\sum_{l=1}^{\infty}a_l\, l!\sum_{k=0}^{\infty}(-1)^k \sum_{J=0}^{\infty}\frac{(b)^{k-J}c^J}{J!(k-J)!}\sum_{r=0}^{\infty}\frac{(k+r)!(-a)^r}{\Gamma[-4-2k-r-J-\alpha-l]}\frac{t^{-2k-r-J-\alpha-l-5}}{r!}
\end{aligned}
$$

$$
\begin{aligned}
y(t) = &\sum_{k=1}^{n} y^{(k-1)}(0)\frac{t^{k-1}}{\Gamma(k)} + y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-2}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-n-1]r!} \\
&+ y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-3}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-n-2]r!} \\
&+ y^{(n+2)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-n-3]r!} \\
&+a\, y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-3}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-n-2]r!} \\
&+a\, y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-n-3]r!} \\
&+b\, y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-n-3]r!} \\
&+\sum_{l=1}^{\infty}a_l\, l!\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-\alpha-l-5}\sum_{J=0}^{\infty}\frac{(b)^{k-J}\left(ct^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{-1})^r}{\Gamma[-2k-r-J-\alpha-l-4]r!}
\end{aligned}
$$

Example 3.4: Let $\alpha = \frac{3}{2}, a = 2, b = 4$ and $c = 5$ in theorem (3.2), then the solution of $y^{\frac{9}{2}}(t) + 2y^{\frac{7}{2}}(t) + 4y^{\frac{5}{2}}(t) + 5y^{\frac{3}{2}}(t) = \sin(2t)$ is

$$y(t) = \sum_{k=1}^{n} y^{(k-1)}(0)\frac{t^{k-1}}{\Gamma(k)} + y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-2}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma[-2k-r-J-n-1]r!}$$

$$+\, y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-3}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma[-2k-r-J-n-2]r!}$$

$$+\, y^{(n+2)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma[-2k-r-J-n-3]r!}$$

$$+2y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-3}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma[-2k-r-J-n-2]r!}$$

$$+2y^{(n+1)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma[-2k-r-J-n-3]r!}$$

$$+4y^{(n)}(0)\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-n-4}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma[-2k-r-J-n-3]r!}$$

$$+\sum_{l=1}^{\infty}(2)_l\, l!\sum_{k=0}^{\infty}(-1)^k\, t^{-2k-l+\frac{7}{2}}\sum_{J=0}^{\infty}\frac{(4)^{k-J}\left(5t^{-1}\right)^J}{J!\Gamma(k-J+1)}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{-r}}{\Gamma\left[-2k-r-J-l+\frac{5}{2}\right]r!}$$

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