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A PATH FOR HORIZING YOUR INNOVATIVE WORK

A NOTE ON ESTIMATION OF VARIANCE COMPONENTS BY MAXIMIZATION

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Abstract: - In this paper, we study the estimation of variance components in the one-way repeated measurements model (one-way- RMM),which contains one within- units factor incorporating one random effects one between- units factor as well as the experimental error term, the estimation is carried out by non-linear maximization which requires the maximum likelihood estimation of variance components. Our aim is to estimate these components according constraints that the variance components are positive, as it was assumed this model, and determining the variance components in the matrix covariance of this model, and then derive the estimators of these components by maximum likelihood mothed.

Keywords: one-way- RMM, variance components, maximum likelihood method

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INTRODUCTION

Repeated measures is a common data structure with multiple measurements on a single unit repeated over time. Multivariate linear models with correlated errors. Repeated measurements analysis is widely used in many fields, for example , in the health and life science, epidemiology, biomedical, agricultural, psychological, educational researches and so on[1]. Repeated measurements occur frequently in observational studies which are longitudinal in nature,and in experimental studies incorporating repeated measures designs.[7]. The problem of variance components estimation from linear model ,it addressed many researchers in different ways, Fisher (1918)work in the development of the style of analysis of variance and the estimation of variance components[4].

Rao and Kleffe(1988) gave methods of the estimation of variance components. Al-Mouel (2004) studied the multivariate repeated measures models and comparison of estimators[2]. Al-Mouel and Wang (2004) presented the sphercity test for one –way multivariate repeated measurements analysis of variance model.They studied the asymptotic expansion of the sphercity test for one –way multivariate repeated measurements analysis of variance model[1]. Mohaisen and Swadi (2014) studied the Bayesian Estimators of the one- way repeated measurements model[6]. In this paper,we study the estimation of variance components in the one-way repeated measurements model (one-way- RMM),which contains one within- units factor incorporating one random effects one between- units factor as well as the experimental error term,the estimation is carried out by non-linear maximization which requires the maximum likelihood estimation.Our aim is to estimate these components according constraints that the variance components are positive ,as it was assumed this model, and determining the variance components in the matrix variation of this model,and then derive the estimators of these components by maximum likelihood method.

2. Variance Components Model

The model

$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \Upsilon_k + e_{ijk}$ (1)

Where

i= 1,2,……,n is an index for experimental unit with group j,

j=1,2,……,q is an index for levels of the between-units factor (Group), k=1,2,……,p is an index for levels of the within- units factor (Time),

y_{ijk} is the respons measurement at time k for unit i within group j ,

μ is the overall mean ,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,

Υ_k is the added effect for time k ,

e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$$\sum_{j=1}^{q} \tau_j = 0 \quad , \quad \sum_{k=1}^{p} \Upsilon_k = 0$$

And we assume that e_{ijk}'s and $\delta_{i(j)}$'s are independent with

$e_{ijk} \sim$ i.i.d $N(0, \sigma_e^2)$, $\delta_{i(j)} \sim$ i.i.d $N(0, \sigma_\delta^2)$, (2)

Can be written as follows
Y=Xβ+Zb+e

Where

Y is the respons vector with $nqp \times 1$ dimention,

X is the desigen matrix with $nqp \times q + p + 1$ dimention,

β is the parametorsvector with $q+p+1 \times 1$ dimention,

Z is the desigen matrix with $nqp \times nq$ dimention

b is the random effects vector with$nq \times 1$dimention ,

e is the random errors with $nqp \times 1$ dimention,

We imposed the following conditions:

$$\left.\begin{array}{l} e \sim \text{i.i.d } N(0, \sigma_e^2 I_{nqp}) \\ b \sim \text{i.i.d } N(0, \mathrm{dig}\sigma_j^2 I_{nq}) \quad j=1,2,\ldots,q \end{array}\right\} \quad (3)$$

cov(b,e)=0

According this condition we have

$Y \sim N(X\beta, V)$

$$V=\sum_{j=1}^{q} \sigma_j^2 Z_j Z_j^T + \sigma_e^2 I_{nqp} \qquad (4)$$

$$=\sum_{j=0}^{q} \sigma_j^2 Z_j {Z_j^T}_j$$

If we assume in (4) $\sigma_0^2 = \sigma_e^2$ then

$$Z_0 Z_0^T = I_{nqp}$$

The parameters $\sigma_0^2, \sigma_1^2, . \dots, \sigma_q^2$ are called the variance components.

Our aim is to estimate these components.

And according constraints

$$\sigma_j^2 \geq 0 \quad , j = 1,2, \dots\dots, q \qquad (5)$$

$$\sigma_0^2 > 0$$

It is known that the constraint (5) leads to the covariance matrix be positive definite matrix (nonsingular).

3. Maximum Likelihood Estimation

Consider

$Y \sim N(X\beta, V)$

Where V is nonsingular

The likelihood function of Y can be written as the following

$$L(\beta,V)=(2\pi)^{-\frac{nqp}{2}} |V|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(Y - X\beta)V^{-1}(Y - X\beta)\right] \qquad (6)$$

$$\ln L(\beta,V)= C-\frac{1}{2}\ln|V| - \frac{1}{2}(Y - X\beta)^T V^{-1}(Y - X\beta) \qquad (7)$$

Where|V| is the determinat of matrix V, C is real constant .

To estimate the parameters β and V by using maximum likelihood mothed of the function L(β,V) according to constraints 5,

We differential (7) with respect to β and equal to zero we get $\hat{\beta}$

$$\frac{\partial \ln L(\beta, V)}{\partial \beta} = X^T V^{-1} Y - (X^T V^{-1} X)\beta \qquad (8)$$

$$\therefore \ \hat{\beta} = (X V^{-1} X)^{-1} X^T V^{-1} Y$$

If V is singular be have for treatment the estimator $\hat{\beta}$,

But our aim in this study was estimate $\sigma_j^2 \quad , (j = 1,2, \ldots \ldots, q)$

To differential (7) with respect to V

Assume that

$$\theta_j = \sigma_j^2 \quad , (j = 1,2, \ldots \ldots, q)$$

Let

$$\theta = (\theta_1, \theta_2, \ldots \ldots, \theta_q)^T$$

We differential (6) with respect to θ

We must prove the following

$$\frac{\partial \ln |V|}{\partial \theta_j} = \operatorname{tr} V^{-1} V_j \qquad (9)$$

$$\frac{\partial V^{-1}}{\partial \theta_j} = - V^{-1} V_j V^{-1} \qquad (10)$$

Where $V_j = \frac{\partial V}{\partial \theta_j}$

Proof 9

$$\frac{\partial \ln|V|}{\partial \sigma_{jk}} = \begin{cases} \sigma^{jj} & \text{if } j = k \\ 2\sigma^{jk} & \text{if } j \neq k \end{cases}$$

$$V^{-1} = (\sigma^{jk})$$

By application of the chain rule

$$\frac{\partial \ln|V|}{\partial\theta} = \sum_j \sum_k \frac{\partial \ln|V|}{\partial\sigma_{jk}} \frac{\partial\sigma_{jk}}{\partial\theta}$$

$$= \sum_j \sum_k \sigma^{jk} \frac{\partial\sigma_{jk}}{\partial\theta} = \mathrm{tr}V^{-1}\frac{\partial V}{\partial\theta}$$

Natural derivation

$$\theta = \left(\theta_1, \theta_2, \ldots\ldots, \theta_q\right)$$

Is adrivation of one of the components of θ_j .

Proof 10

Since $V^{-1}V = I$

$$\frac{\partial V^{-1}}{\partial\theta_j} V + V^{-1}\frac{\partial V^{-1}}{\partial\theta_j} = 0 \qquad (11)$$

By using the equations (7) , (9) we have q of equations, produced from

$$\frac{\partial \ln L(\beta,V)}{\partial\theta_j} = -\frac{1}{2}\mathrm{tr}\, V^{-1} V_j + \frac{1}{2}WV^{-1} V_j V^{-1} W \ , j=1,2,\ldots,q \qquad (12)$$

W=Y – Fβ

We can solve these equations numerically , for purposes of calculation possible in writing

$$\frac{\partial \ln L(\beta,V)}{\partial\theta_j} = -\frac{1}{2}\mathrm{tr}\widehat{G}\,\widehat{V}_j = 0 \qquad (13)$$

G=$V^{-1} - SS^T$

S=$V^{-1}W$

These equations can be solved iteratively only and it includes one of the modalities of the solution information matrix, we get,

$$\frac{\partial^2 V^{-1}}{\partial\theta_j\,\partial\theta_k} = B_{jk}V^{-1} + B_{kj}V^{-1} - R_{jk} \qquad (14)$$

Therefore

$V_{jk} = \frac{\partial^2 V}{\partial\theta_j \, \partial\theta_k}$

$R_{jk} = V_{jk}V^{-1}$

$B_{jk} = V^{-1}V_j V^{-1}V_k$

Since

$\frac{\partial}{\partial\theta_j} \mathrm{tr}\, C\, V = \mathrm{tr}\, C\, V_j$, for any C constants matrix,

Therefore from (8) gets

$\frac{\partial^2 \ln V}{\partial\theta_j \, \partial\theta_k} = \mathrm{tr}(B_{jk} - R_{jk})$

we differential (14) with respect to θ_k and using (9) ,(10) we fined

$$\frac{2\,\partial^2 \ln L(\beta,V)}{\partial\theta_j \, \partial\theta_k} = -\mathrm{tr}(R_{jk} - B_{jk}) \text{ -}W^{-1}(B_{jk} - B_{kj} - R_{jk})V^{-1}W \qquad (15)$$

Since

E(W)=0

E(WCW)=tr VC

$$E\left(\frac{-\partial^2 \ln L(\beta,V)}{\partial\theta_j \, \partial\theta_k}\right) = \frac{1}{2} \mathrm{tr}\, B_{jk} \qquad (16)$$

Assume that

$$A=\frac{1}{2} \mathrm{tr}\, B_{jk} \qquad (17)$$

We have

$$\frac{\partial^2 \ln L(\beta,V)}{\partial\beta \, \partial\theta_j} = -F^T V^{-1}V_k W \qquad (18)$$

$$E\left(\frac{\partial^2 \ln L(\beta,V)}{\partial\beta \, \partial\theta_j}\right) = 0 \qquad (19)$$

Because

E(W)=E(Y-Fβ) $= 0$

Also

$\frac{\partial^2 \ln L(\beta,V)}{\partial \beta^2}$= $FV^{-1}F$

The information matrix for the (β, θ) will be

$$M(\beta,\theta) = \begin{bmatrix} FV^{-1}F & 0 \\ 0 & A \end{bmatrix} \quad (20)$$

The variance matrix of estimator $\hat{\beta}$ is

Cov($\hat{\beta}$) $= (F\ V^{-1}F)^{-1}$

The variance matrix of estimator $\hat{\theta}$ is

Cov($\hat{\theta}$) $=\ A^{-1}$

Where

A =$\frac{1}{2}$ tr β_{jk}

The equations (8) ,(13) can be resolved through repetition

If we assume that

θ_0 : Initial value of the parameter θ

From (8) we can get the value of β

We use Newton-Raphson mothed,

To update θ in every step using

$$\theta_{m+1} = \theta_m + A_m^{-1}(\delta_m) \quad (21)$$

Where

A_m the matrix A in θ_m

δ_m = -$\frac{1}{2}$ tr G V_j

Is a vector of derivatives defined by (13),

Calculated at θ_m and β_m

To update in every step using (8)

To estimate θ_j only .

4. CONCLUSION

The conclusions which are obtained throughout this work are given as follows :

1- The maximum likelihood estimetors for the parameters β is

$$\hat{\beta} = (XV^{-1}X)^{-1}\, X^T V^{-1} Y\,,$$

2- The variance matrix of estimator $\hat{\beta}$ is Cov($\hat{\beta}$) $= (F\, V^{-1} F)^{-1}$

3- The variance matrix of estimator $\hat{\theta}$ is Cov($\hat{\theta}$) $= \ A^{-1}$, A $= \frac{1}{2}\,\mathrm{tr}\ \beta_{jk}$

4- The information matrix for the (β, θ) is M$(\beta, \theta) = \begin{bmatrix} FV^{-1}F & 0 \\ 0 & A \end{bmatrix}$

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