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PARTICULAR SOLUTION OF LINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS BY INVERSE OPERATORS

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Abstract: This paper adopts the inverse fractional differential operators method for obtaining the explicit particular solution to a linear sequential fractional differential equation (LSFDE), involving Jumarie's modification of Riemann–Liouville derivative, with constant coefficients. This method depend on the classical inverse differential operators method and it is independent of the integral transforms. Several examples are then given to demonstrate the validity of our main results.

Keywords: Fractional differential equations, Riemann–Liouville derivative, Jumarie's fractional derivation, Inverse differential operators, Inverse fractional differential operators.

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1. INTRODUCTION

Fractional differential equations (FDEs) occur in numerous complex systems in life science, such as rheology, viscoelasticity, porous media, electrochemistry, electromagnetism, dynamics of earthquakes, geology, viscoelastic materials, bioengineering, signal processing, optics, biosciences, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, converters, electromagnetic waves, control of power electronic, fluid mechanics, chaotic dynamics, proteins, polymer physics, statistical physics, polymer science, electrochemistry, thermodynamics, neural networks, ,and many more (Diethelm and Freed, 1999; Gaul et al., 1991; Glockle and Nonnenmacher, 1995; Magin, 2006; Metzler et al., 1995; Sheng, 2011; Zaslavsky, 2005). In recent years, there has been a major development in the methods of solving FDEs (Abbas, et al., 2012; Agarwal, et al., 2009, 2010, 2011; Arshad and Lupulescu, 2011; Diethelm, 2010; Diethelm and Ford, 1999, 2002; Eidelman and Kochubei, 2004; Jiang and Ding 2013; Jiang, 2013; Khalaf 2016, Kilbas et al. 2006, Khudair 2013; Khudair et. al. 2016; LI and WANG 2013; Liu and Li, 2013; Mophou, 2010; Miller and Ross, 1993; Podlubny, 1999; Sabatier et al. , 2007; Samko et al.,1993; Sheng et al., 2011; Vong, 2013; Zaslavsky, 2005; Zhai et al., 2013) and the references therein. A wide description of the problem of the existence and uniqueness of solutions of Cauchy-type problems for fractional order differential equations together with its applications can be found in the literature (Abbas et al, 2012; Diethelm, 2010; Kilbas et al., 2006; Magin, 2006; Miller and Ross, 1993; Podlubny, 1999; Sabatier et al. , 2007; Samko et al.,1993; Sheng et al., 2011).

Motivated and inspired by the on-going research in this field, we will consider the following non-homogeneous linear fractional differential equation with constant coefficient

$$(\mathcal{D}_x^{n\alpha} + a_1\mathcal{D}_x^{(n-1)\alpha} + a_2\mathcal{D}_x^{(n-2)\alpha} + \cdots + a_{n-1}\mathcal{D}_x^{\alpha} + a_n)y(x) = Q(x) \tag{1.1}$$

where $q = \dfrac{1}{\alpha}$ is integer number, $a_k,\ k = 1,2,\ldots,n$ are real constant , $\mathcal{D}_x^{n\alpha} = \underbrace{D_x^{\alpha} D_x^{\alpha} \cdots D_x^{\alpha}}_{n-\text{times}}$

and D_x^{α} denotes Jumarie's fractional derivation (Jumarie, 1993, 2006, 2007, 2009, 2010) , which is a modified Riemann–Liouville derivative defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\zeta)^{-\alpha} (f(\zeta) - f(0))d\zeta, \qquad 0 < \alpha < 1 \tag{1.2}$$

And

$$D_x^{\alpha}f(x) = \frac{d^n}{dx^n}(D^{(\alpha-n)}f(x)), \qquad n < \alpha < n+1, \; n \geq 1 \tag{1.3}$$

Eq.(1) is called fractional linear differential equation with constant coefficients of order (n,q), or more briefly, a fractional differential equation of order (n,q) (Miller and Ross, 1993) If $\alpha = 1$, then Eq.(1.1) become n^{th} order ordinary differential equations.

This paper is organised as follows: Sections 2 presents Jumarie's Modification of Riemann–Liouville Derivative and their main properties. In section 3, we study some properties of linear fractional differential operators with constant coefficients. In section 4, we adopt the method of inverse fractional differential operators to find the particular solution to non-homogeneous LSFDE with constant coefficients while in section 5, several examples are given to demonstrate the validity of our main results.

2. JUMARIE' MODIFICATION OF RIEMANN–LIOVILLE DERIVATIVE

The fractional derivative have deferent definitions (Kilbas et al., 2006; Miller and Ross, 1993; Podlubny, 1999; Sabatier et al. , 2007; Samko et al.,1993; Sheng et al., 2011), and exploiting any of them depends on the boundary conditions and the specifics of the considered physical systems and processes. The first definition of fractional derivative which has been proposed in the literature is the so-called Riemann-Liouville definition which reads as follows

Definition 2.1 (Riemann-Liouville derivative) Let $f(x) := R \to R$ be a continuous function then the fractional derivative of order α is defined by

$$D_x^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-\zeta)^{-\alpha-1}f(\zeta)d\zeta \quad , \quad \alpha < 0 \tag{2.1}$$

and

$$D_x^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\zeta)^{n-\alpha-1}f(\zeta)d\zeta, \; n < \alpha < n+1 \tag{2.2}$$

It is well known that the fractional derivative, in the sense of Riemann-Liouville definition of fractional derivative , of a constant is not zero. This encourage Caputo to introduce Caputo derivative such that the fractional derivative of a constant is zero (Caputo, 2008).

Definition 2.2 (Caputo derivative) Let $f(x):=R\to R$ be a continuous function then the fractional derivative of order α is defined by

$$D_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-\zeta)^{n-\alpha-1}\frac{d^nf(\zeta)}{d\zeta^n}d\zeta,\quad n<\alpha<n+1 \tag{2.3}$$

With Caputo definition, a fractional derivative would be defined for differentiable functions only. In order to deal with non-differentiable functions, Jumarie have recently proposed a modification of the Riemann-Liouville definition (Jumarie, 1993, 2006, 2007, 2009, 2010). This fractional derivative provides a Taylor's series of fractional order for non differentiable functions.

Definition 2.3 (Jumarie's modification of Riemann-Liouville derivative): Let $f(x):=R\to R$ be a continuous function then the fractional derivative of order α is defined by

$$D_x^{\alpha}f(x)=\frac{1}{\Gamma(-\alpha)}\int_0^x(x-\zeta)^{-\alpha-1}(f(\zeta)-f(0))d\zeta\,,\alpha<0 \tag{2.4}$$

and

$$D_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x(x-\zeta)^{n-\alpha-1}(f(\zeta)-f(0))d\zeta,\quad n<\alpha<n+1 \tag{2.5}$$

He, et al, (He et al., 2012) introduce the geometrical explanation of fractional complex transform and derivative chain rule for fractional calculus in the sense of Jumarie's modification of Riemann-Liouville derivative. Remark the main difference between definition (2.2) and definition (2.5). The second one involves the constant $f(0)$ while the first one does not. Also, the fractional Riemann–Liouville derivative of a constant is not zero while the fractional Jumarie derivative of a constant is zero. In the rest of the paper, D_x^{α} will be used to refer to Jumarie's modification of Riemann-Liouville derivative.

Definition 2.4 (Principle of Derivative increasing orders) : The functional derivative of fractional $D_x^{\alpha+\beta}$ expressed in terms of D_x^{α} and D_x^{β} is defined by the equality $D_x^{\alpha+\beta}f(x)=D_x^{\max(\alpha,\beta)}(D_x^{\min(\alpha,\beta)}f(x))$.

Proposition 2.5: Assume that the continuous function $f(x) := R \to R$ has a fractional derivative of order αk for any positive integer k and $0 < \alpha < 1$, then the following equality holds (Jumarie, 2009).

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(\alpha k+1)} f^{(\alpha k)}(x), \quad 0 < \alpha \le 1 \tag{2.6}$$

where $f^{(\alpha k)}(x)$ is the fractional Jumarie derivative of order αk of $f(x)$. Formally, Eq.(2.6) can be written $f(x+h) = E_\alpha(h^\alpha D_x^\alpha) f(x), \quad 0 < \alpha \le 1$, where $E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(\alpha k+1)}$.

Corollary 2.6:The following equalities hold (Jumarie, 2009)., which are

$$D^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha}, \quad \gamma > 0 \tag{2.7}$$

or, what amounts to the same (we set $\alpha = n + \theta$)

$$D^{n+\theta} x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta)x^{\gamma-n-\theta}, \quad 0 < \theta < 1 \tag{2.8}$$

$$D_x^\alpha(u(x)v(x)) = D_x^\alpha u(x)\, v(x) + u(x)\, D_x^\alpha v(x) \tag{2.9}$$

$$D_x^\alpha(f(u(x))) = \frac{df(u)}{du} D_x^\alpha u(x) \tag{2.10}$$

$$D_x^\alpha(f(u(x))) = D_u^\alpha f(u)\, D_x^\alpha\left(\frac{du(x)}{dx}\right) \tag{2.11}$$

Lemma 2.7: The following various formulae are hold (Jumarie, 2009).

1. $E_\alpha(x^\alpha y^\alpha) = (E_\alpha(y^\alpha))^x$ (2.12)

2. $E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) + E_\alpha(\lambda y^\alpha)$ (2.13)

3. $D_x^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$ (2.14)

4. $D_x^\alpha E_\alpha(\lambda u^\alpha(x)) = E_\alpha(u^\alpha) D_x^\alpha(u'(x))$ (2.15)

5. $E_\alpha(ix) = \cos_\alpha x + i\sin_\alpha x$ (2.16)

6. $E_\alpha(x) = \cosh_\alpha x + \sinh_\alpha x$ (2.17)

7. $D_x^\alpha \cos_\alpha x^\alpha = -\sin_\alpha x^\alpha,\ D_x^\alpha \sin_\alpha x^\alpha = \cos_\alpha x^\alpha$ (2.18)

8. $D_x^\alpha \cosh_\alpha x^\alpha = -\sinh_\alpha x^\alpha,\ D_x^\alpha \sinh_\alpha x^\alpha = \cosh_\alpha x^\alpha$ (2.19)

9. $D_x^\alpha E_\alpha(\lambda x) = \lambda\alpha^{-\alpha} x^{1-\alpha} E_\alpha(\lambda x)$ (2.20)

3. SOME PROPERTIES OF LINEAR FRACTIONAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS:

Consider the following linear non homogeneous LSFDE with constant coefficients of order (n,q)

$$(\mathcal{D}_x^{n\alpha} + a_1\mathcal{D}_x^{(n-1)\alpha} + a_2\mathcal{D}_x^{(n-2)\alpha} + \cdots + a_{n-1}\mathcal{D}_x^{\alpha} + a_n)y(x) = Q(x) \tag{3.1}$$

where $\alpha = \frac{1}{q}$ is constant rational number, $a_k,\ k = 1,2,\ldots,n$ are real constant ,

$$\mathcal{D}_x^{n\alpha} = \underbrace{D_x^\alpha D_x^\alpha \cdots D_x^\alpha}_{n-\text{times}}.$$

Rewrite Eq.(3.1) in the form

$$P(\mathcal{D}_x^\alpha)y(x) = Q(x) \tag{3.2}$$

where $P(\mathcal{D}_x^\alpha)$ is a linear fractional differential operator.

Lemma(3.1):(The Exponential Mittag-Leffler shift)

$E_\alpha(\lambda x^\alpha)P(\mathcal{D}_x^\alpha)y(x) = P(\mathcal{D}_x^\alpha - \lambda)E_\alpha(\lambda x^\alpha)y(x)$ where $E_\alpha(u) = \sum_{k=0}^{\infty}\frac{u^k}{\Gamma(\alpha k+1)}$ is the Exponential Mittag-Leffler function.

Proof:

Consider the effect of the operator $\mathcal{D}_x^\alpha - \lambda$ on the product of $E_\alpha(\lambda x^\alpha)$ and a function $y(x)$, one can have

$$(\mathcal{D}_x^\alpha - \lambda)E_\alpha(\lambda x^\alpha)y(x) = \mathcal{D}_x^\alpha E_\alpha(\lambda x^\alpha)y(x) - \lambda E_\alpha(\lambda x^\alpha)y(x)$$

$$= (\mathcal{D}_x^\alpha E_\alpha(\lambda x^\alpha))y(x) + E_\alpha(\lambda x^\alpha)\mathcal{D}_x^\alpha y(x) - \lambda E_\alpha(\lambda x^\alpha)y(x)$$

$$= \lambda E_\alpha(\lambda x^\alpha)y(x) + E_\alpha(\lambda x^\alpha)\mathcal{D}_x^\alpha y(x) - \lambda E_\alpha(\lambda x^\alpha)y(x)$$

$$= E_\alpha(\lambda x^\alpha)\mathcal{D}_x^\alpha y(x).$$

And

$$(\mathcal{D}_x^\alpha - \lambda)^2 E_\alpha(\lambda x^\alpha)y(x) = (\mathcal{D}_x^\alpha - \lambda)(\mathcal{D}_x^\alpha - \lambda)E_\alpha(\lambda x^\alpha)y(x)$$

$$= (\mathcal{D}_x^\alpha - \lambda)(E_\alpha(\lambda x^\alpha)\mathcal{D}_x^\alpha y(x))$$

$$= E_\alpha(\lambda x^\alpha)\mathcal{D}_x^\alpha \mathcal{D}_x^\alpha y(x)$$

$$= E_\alpha(\lambda x^\alpha)\mathcal{D}_x^{2\alpha} y(x).$$

Repeating the operation, one have

$$(\mathcal{D}_x^\alpha - \lambda)^k = E_\alpha(\lambda x^\alpha)\mathcal{D}_x^{k\alpha} y(x), \quad k = 1, 2, \ldots \tag{3.3}$$

Using the linearity of fractional differential operators, we conclude that when $P(\mathcal{D}_x^\alpha)$ is a polynomial in $\mathcal{D}_x^\alpha$ with constant coefficients, then

$$E_\alpha(\lambda x^\alpha)P(\mathcal{D}_x^\alpha)y(x) = P(\mathcal{D}_x^\alpha - \lambda)E_\alpha(\lambda x^\alpha)y(x) \tag{3.4}$$

As a direct computation, one has the following :

Lemma 3.2: The following various formulae are hold

1. $P(\mathcal{D}_x^\alpha)E_\alpha(\lambda x^\alpha) = P(\lambda)E_\alpha(\lambda x^\alpha)$ (3.5)

2. $P(\mathcal{D}_x^{2\alpha})\cos_\alpha bx^\alpha = P(-b^2)\cos_\alpha bx^\alpha$, (3.6)

3. $P(\mathcal{D}_x^{2\alpha})\sin_\alpha bx^\alpha = P(-b^2)\sin_\alpha bx^\alpha$, (3.7)

4. $P(\mathcal{D}_x^{2\alpha})\cosh_\alpha bx^\alpha = P(-b^2)\cosh_\alpha bx^\alpha$, (3.8)

5. $P(\mathcal{D}_x^{2\alpha})\sinh_\alpha bx^\alpha = P(-b^2)\sinh_\alpha bx^\alpha$, (3.9)

4. PARTICULAR SOLUTION 0F NONHOMOGENEOUS LSFDE WITH CONSTANT COEFFICIENTS BY USING INVERSE FRACTIONAL DIFFERENTIAL OPERATORS

In seeking a particular solution of Eq.(3.2), it is natural to write

$$y(x) = \frac{1}{P(\mathcal{D}_x^\alpha)}Q(x) \tag{4.1}$$

and to try to define an operator $\frac{1}{P(\mathcal{D}_x^\alpha)}$ so that the function $y(x)$ of Eq.(4.1) will have meaning and will satisfy Eq.(3.2).

Instead of building a theory of such inverse fractional differential operators, we shall adopt the following method of attack. Purely formal (unjustified) manipulations of the symbols will be performed, thus leading to a tentative evaluation of $\frac{1}{P(\mathcal{D}_x^\alpha)}Q(x)$. After all, the only thing that we require of evaluation is that

$$P(\mathcal{D}_x^\alpha)\frac{1}{P(\mathcal{D}_x^\alpha)}Q(x) = Q(x) \tag{4.2}$$

Hence the proof will be placed on a direct verification of the Eq.(4.2) in each instance.
By using lemma(3.2), one has the following :

Lemma 4.1: The following various formulae are hold

1. $\frac{1}{P(\mathcal{D}_x^\alpha)}E_\alpha(\lambda x^\alpha) = \frac{1}{P(\lambda)}E_\alpha(\lambda x^\alpha)$, such that $P(\lambda) \neq 0$ (4.3)

2. $\frac{1}{P(\mathcal{D}_x^{2\alpha})}\sin_\alpha bx^\alpha = \frac{1}{P(-b^2)}\sin_\alpha bx^\alpha$, such that $P(-b^2) \neq 0$ (4.4)

3. $\frac{1}{P(\mathcal{D}_x^{2\alpha})}\cos_\alpha bx^\alpha = \frac{1}{P(-b^2)}\cos_\alpha bx^\alpha$, such that $P(-b^2) \neq 0$ (4.5)

4. $\frac{1}{P(\mathcal{D}_x^{2\alpha})}\sinh_\alpha bx^\alpha = \frac{1}{P(-b^2)}\sinh_\alpha bx^\alpha$, such that $P(-b^2) \neq 0$ (4.6)

5. $\frac{1}{P(\mathcal{D}_x^{2\alpha})}\cosh_\alpha bx^\alpha = \frac{1}{P(-b^2)}\cosh_\alpha bx^\alpha$, such that $P(-b^2) \neq 0$ (4.7)

6. $\frac{1}{P(\mathcal{D}_x^{\alpha})}E_\alpha(\lambda x^\alpha)f(x) = E_\alpha(\lambda x^\alpha)\frac{1}{P(\mathcal{D}_x^{\alpha}+\lambda)}f(x),$ (4.8)

Lemma 4.2: Let $P(m)$ be a polynomial of degree n and its roots are $\beta_k, k = 1, 2, \ldots, n$,i.e. $P(m) = \prod_{k=1}^{n}(m-\beta_k)$ then

$$\frac{1}{P(\mathcal{D}_x^{\alpha})}f(x) = \frac{\prod_{k=1}^{n}(\mathcal{D}_x^{(q-1)\alpha} + \beta_k\mathcal{D}_x^{(q-2)\alpha} + \beta_k^2\mathcal{D}_x^{(q-3)\alpha} + \cdots + \beta_k^{q-1})}{\prod_{k=1}^{n}(D-\beta^q{}_k)}f(x) \tag{4.9}$$

where $D = \frac{d}{dx}$ and $q = \frac{1}{\alpha}$ is integer number.

Proof: Note that

$$\prod_{k=1}^{n}(D-\beta^q{}_k) = \prod_{k=1}^{n}(\mathcal{D}_x^{q\alpha} - \beta_k^q)$$

$$= \prod_{k=1}^{n}(\mathcal{D}_x^{\alpha} - \beta_k)(\mathcal{D}_x^{(q-1)\alpha} + \beta_k\mathcal{D}_x^{(q-2)\alpha} + \beta_k^2\mathcal{D}_x^{(q-3)\alpha} + \cdots + \beta_k^{q-1})$$

$$= \left(\prod_{k=1}^{n}(\mathcal{D}_x^{\alpha} - \beta_k)\right)\left(\prod_{k=1}^{n}(\mathcal{D}_x^{(q-1)\alpha} + \beta_k\mathcal{D}_x^{(q-2)\alpha} + \beta_k^2\mathcal{D}_x^{(q-3)\alpha} + \cdots + \beta_k^{q-1})\right)$$

$$= P(\mathcal{D}_x^{\alpha})\left(\prod_{k=1}^{n}(\mathcal{D}_x^{(q-1)\alpha} + \beta_k \mathcal{D}_x^{(q-2)\alpha} + \beta_k^2 \mathcal{D}_x^{(q-3)\alpha} + \cdots + \beta_k^{q-1})\right)$$

So, one can have

$$\prod_{k=1}^{n}(D - \beta^q{}_k)f(x) = P(\mathcal{D}_x^{\alpha})\left(\prod_{k=1}^{n}(\mathcal{D}_x^{(q-1)\alpha} + \beta_k \mathcal{D}_x^{(q-2)\alpha} + \beta_k^2 \mathcal{D}_x^{(q-3)\alpha} + \cdots + \beta_k^{q-1})\right)f(x)$$

This imply Eq.(4.9).

Remark that lemma(4.2) is very important specially, when the right member of Eq.(3.2) is $e^{ax}, \cos(ax), \cosh(ax), \sin(ax), \sinh(ax), x^m$ or any combination of these functions. In fact, lemma (4.2) will be used the classical inverse differential operator in order to compute the inverse fractional differential operator. In the next section, we adopt several examples to illustrate the advantage of Method.

5. ILLUSTRARED EXAMPLES:

Example 1 : we consider the nonhomogeneous fractional differential equation

$$(\mathcal{D}^{\frac{1}{2}} - 2)y(x) = e^x \tag{5.1}$$

Clearly, the auxiliary equation is $p(m) = m - 2 = 0$ and its root is $m = 2$. Then by using lemma (4.2), one have

$$y_p(x) = \frac{\mathcal{D}^{\frac{1}{2}} + 2}{D - 4} e^x$$

$$y_p(x) = (\mathcal{D}^{\frac{1}{2}} + 2)\frac{1}{-3}e^x \qquad \text{since} \quad \frac{1}{D-4}e^x = \frac{1}{-3}e^x$$

$$y_p(x) = \frac{-1}{3}\mathcal{D}^{\frac{1}{2}} e^x - \frac{2}{3}e^x$$

$$y_p(x) = \frac{-1}{3}\mathcal{D}^{\frac{1}{2}} \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)} - \frac{2}{3}e^x$$

$$y_p(x) = \frac{-1}{3}\sum_{k=1}^{\infty}\frac{x^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2})} - \frac{2}{3}e^x$$

$$y_p(x) = \frac{-1}{3}\sum_{j=0}^{\infty}\frac{x^{j+\frac{1}{2}}}{\Gamma(j+\frac{3}{2})} - \frac{2}{3}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+1)}$$

$$y_p(x) = \frac{-1}{2}E_{\frac{1}{2}}(x^{\frac{1}{2}}) - \frac{1}{6}E_{\frac{1}{2}}(-x^{\frac{1}{2}}) \tag{5.2}$$

It is easily verify that $y_p(x)$ in Eq.(5.2) is particular solution to Eq.(5.1)

Example 2 : we consider the homogeneous fractional differential equation

$$(\mathcal{D} + \mathcal{D}^{\frac{1}{2}} - 2)y(x) = \cos(x) \tag{5.3}$$

Clearly, the auxiliary equation is $p(m) = m^2 + m - 2 = 0$ and its roots are $m = 1, -2$. Then by using lemma (4.2), one have

$$y_p(x) = \frac{(\mathcal{D}^{\frac{1}{2}} + 1)(\mathcal{D}^{\frac{1}{2}} - 2)}{(D^2 - 4)(D^2 - 1)}\cos(x)$$

$$y_p(x) = \frac{(\mathcal{D}^{\frac{1}{2}} + 1)(\mathcal{D}^{\frac{1}{2}} - 2)}{10}\cos(x) \quad \text{since} \quad \frac{1}{(D^2 - 4)(D^2 - 1)}\cos(x) = \frac{\cos(x)}{10}$$

$$y_p(x) = \frac{(D - \mathcal{D}^{\frac{1}{2}} - 2)}{10}\cos(x)$$

$$y_p(x) = \frac{-\sin(x) - \mathcal{D}^{\frac{1}{2}}\cos(x) - 2\cos(x)}{10}$$

$$y_p(x) = \frac{-\sin(x) - 2\cos(x) - \mathcal{D}^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(2k+1)}}{10}$$

$$y_p(x) = \frac{-\sin(x) - 2\cos(x) - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-\frac{1}{2}}}{\Gamma(2k+\frac{1}{2})}}{10} \qquad (5.4)$$

is easily verify that $y_p(x)$ in Eq.(5.4) is particular solution to Eq.(5.3).

Example 3 : we consider the homogeneous fractional differential equation

$$(\mathcal{D}^{\frac{3}{2}} + 2\mathcal{D}^{\frac{1}{2}} - 2)y(x) = E_{\frac{1}{2}}(x^{\frac{1}{2}}) \qquad (5.5)$$

By using lemma (4.1), one can have

$$y_p(x) = \frac{1}{(\mathcal{D}^{\frac{3}{2}} + 2\mathcal{D}^{\frac{1}{2}} - 2)} E_{\frac{1}{2}}(x^{\frac{1}{2}})$$

$$y_p(x) = \frac{1}{(1+2-2)} E_{\frac{1}{2}}(x^{\frac{1}{2}})$$

$$y_p(x) = E_{\frac{1}{2}}(x^{\frac{1}{2}}) \qquad (5.6)$$

It is easily verify that $y_p(x)$ in Eq.(5.6) is particular solution to Eq.(5.5).

Example 4 : we consider the homogeneous fractional differential equation

$$(\mathcal{D} + 2\mathcal{D}^{\frac{1}{2}} - 3)y(x) = E_{\frac{1}{2}}(x^{\frac{1}{2}}) \qquad (5.7)$$

By using lemma (4.1), one can have

$$y_p(x) = \frac{1}{(\mathcal{D} + 2\mathcal{D}^{\frac{1}{2}} - 3)} E_{\frac{1}{2}}(x^{\frac{1}{2}})$$

$$y_p(x) = \frac{1}{(\mathcal{D}^{\frac{1}{2}} + 3)(\mathcal{D}^{\frac{1}{2}} - 1)} E_{\frac{1}{2}}(x^{\frac{1}{2}})$$

$$y_p(x) = \frac{1}{4(\mathcal{D}^{\frac{1}{2}} - 1)} E_{\frac{1}{2}}(x^{\frac{1}{2}})$$

$$y_p(x) = \frac{E_{\frac{1}{2}}(x^{\frac{1}{2}})}{4\mathcal{D}^{\frac{1}{2}}} = \frac{E_{\frac{1}{2}}(x^{\frac{1}{2}})\mathcal{D}^{\frac{1}{2}}}{4}\frac{1}{D}$$

$$y_p(x) = \frac{E_{\frac{1}{2}}(x^{\frac{1}{2}})\mathcal{D}^{\frac{1}{2}}x}{4} \quad \text{since} \quad \frac{1}{D} = x$$

$$y_p(x) = \frac{E_{\frac{1}{2}}(x^{\frac{1}{2}})\sqrt{x}}{2\sqrt{\pi}} \tag{5.8}$$

is easily verify that $y_p(x)$ in Eq.(5.8) is particular solution to Eq.(5.7).

7. Conclusion:

Depending on the roots of the characteristic polynomial of the corresponding homogeneous equation, the inverse fractional differential operators method is established to obtain an explicit particular solution to a linear sequential fractional differential equation (LSFDE), involving Jumarie's modification of Riemann–Liouville derivative, with constant coefficients. This method is independent of the integral transforms but it is applicable when, and only when, the right member of the Eq.(1) is e^{ax}, $\cos(ax)$, $\cosh(ax)$, $\sin(ax)$, $\sinh(ax)$, x^a, $E_\alpha(ax)$, $E_b(ax^b)$ or any combination of these functions.

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