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A PATH FOR HORIZING YOUR INNOVATIVE WORK

TUTORIAL IN PANEL DATA MODEL

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Abstract: - The contribution of this paper is to provide of theoretical results for panel data model, We consider the random effect panel data model. Maximum likelihood method is employed to making inferences on the model, and we prove some properties about the parameters estimators.

Keywords: Panel Data Model, Maximum Likelihood Method, Best Linear Unbiased Estimator (BLUE), Sufficiency, Efficiency, Likelihood Ratio Test.

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INTRODUCTION

The statistical data is important to study most of phenomena as economical, social, psychological phenomena. etc. The analysis of this data via the statistical methods gives the researcher or the decision maker more information about the studied phenomenon to make the suitable decision. The data availability needs to limit a mathematic model which represents them by the researcher and to put in consideration the type of the available data. One of the these data is panel which can be defined (that they are repeated measurements to the studied phenomena for N from the cross section and for T from the time series) which can be represented by one of the model (fixed effect model or random effect model).

One of the aims of science is to describe and predict events in the world in which we live. One way this is accomplished is by finding a formula or equation that relates quantities in the real world,[2].

In spite of the availability of highly innovative tools in statistics, the main tool of the applied statistician remains the linear model. The linear model involves the simplest and seemingly most restrictive statistical properties: independence, normality, constancy of variance, and linearity . However, the model and the statistical methods associated with it are surprisingly versatile and robust. More importantly, mastery of linear model is a prerequisite to work with advanced statistical tools because most advanced tools are generalizations of the linear model. The linear model is thus central to the training of any statistician, applied or theoretical, [6] .

In applied sciences, ones is often confronted with the collection of correlated data. This generic term embraces a multitude of data structures, such as multivariate observations, clustered data , repeated measurements , longitudinal data , and spatially correlated data .

Panel (or longitudinal) data are cross – sectional and time – series. There are multiple entities, each of which has repeated measurements at different time periods. Panel data have a cross – sectional (entity or subject) variable and a time series variable. In Stata, this arrangement is called the long form (as opposed to the wide form). While the long form has both group (individual level) and time variables, the wide form includes either group or time variables. Panel data usually give the researcher a large number of data points, increasing the degrees of freedom and reducing the collinearity among explanatory variables. Panel data models have becomes increasingly popular among applied researchers due to their heightened capacity for capacity for capturing the complexity of human behaviour as compared to cross-sectional or time –series data models. As a consequence , more and richer panel data sets also have become increasingly available, [1],[3],[7],[8],[9],[10].

The problem we are interest in this paper is that of estimating the value of an $(K+1)$ −dimensional vector of parameters θ as well as the values of the variances parameters σ_{ε}^2 and σ_1^2 for the panel data model. Two important problems in statistical inference are estimation and tests of hypotheses are to be the subject of many books for example references [4] and [5]. In the text context of this paper we are only interested in the maximum likelihood approach.

The contribution of this paper is to provide of theoretical results for panel data model, We consider the random effect panel data model . Maximum likelihood method is employed to making inferences on the model, and we prove some properties about the likelihood estimators and likelihood ratio test statistics are given here.

2. Panel Data Model

Consider the model:

$$Y_{it} = \mu + \sum_{j=1}^{K} \beta_j X_{jit} + \varepsilon_{it}, i = 1, \dots, N, t = 1, \dots, T, \quad (1)$$

Where, Y_{it} the value of response variable for i^{th} unit at time t , $X_{jit}$the explanatory variables , $\mu, \beta_j, j = 1, \dots, K$ are fixed parameters and ε_{it} is an error term with $\varepsilon_{it} \overset{iid}{\sim} N(0, \sigma_{\varepsilon}^2)$.

Now, if the parameter μ is specified as:

$$\mu = \beta_0 + u_i , \quad (2)$$

Where, $u_i \sim N(0, \sigma_u^2)$, then, the model (1) is

$$Y_{it} = \beta_0 + \sum_{j=1}^{K} \beta_j x_{jit} + u_i + \varepsilon_{it} . \quad (3)$$

The model (3) is rewrite as follows:

$$Y_{it} = \beta_0 + \sum_{j=1}^{K} \beta_j x_{jit} + \omega_{it} , \quad (4)$$

Where, $\omega_{it} = u_i + \varepsilon_{it}$, $\omega_{it} \sim N(0 , \sigma_{\omega}^2)$, $\sigma_{\omega}^2 = \sigma_{\varepsilon}^2 + \sigma_u^2$, thus by using matrix notation the model (4) is

$$Y = F\theta + \omega , \quad (5)$$

where, $F = [e\,,X]$, $e = [\,1,1,\dots,1]^T$ has length NT , $Y = [\,Y_{11},\dots,Y_{1T},Y_{21},\dots Y_{2T},\dots,Y_{N1},\dots,Y_{NT}\,]^T$ has length NT , $X = [X_1,X_2,\dots,X_N\,]^T$ is a $NT \times K$ design matrix of fixed effects , $\theta = [\beta_0,\beta_1,\dots,\beta_K]^T$ has length $K+1$, and

$\omega = [\,\omega_{11},\dots,\omega_{1T},\omega_{21},\dots\omega_{2T},\dots,\omega_{N1},\dots,\omega_{NT}\,]^T$ has length NT . From model (5) , we have $Y \sim N\,(\,F\,\theta\,,\Psi\,)$, where $\Psi = E\,(\omega\omega^T\,) = I_N \otimes (\sigma_\varepsilon^2\, I_t + \sigma_u^2\, ee^T)$

$$= \sigma_\varepsilon^2 (I_N \otimes I_t) + \sigma_u^2 (I_N \otimes ee^T),$$

replace I_T by ($E_T + J_T$) and ee^T by $T\,J_T$, where $J_T = \frac{1}{T} ee^T$ and $E_T = I_T - J_T$, then

$$\Psi = \sigma_\varepsilon^2\,[\,I_N \otimes (\,E_T + J_T\,)\,] + \sigma_u^2\,(\,I_N \otimes T\,J_T\,)$$

$$= \sigma_\varepsilon^2\,(I_N \otimes E_T\,) + \sigma_\varepsilon^2\,(I_N \otimes J_T\,) + T\sigma_u^2\,(\,I_N \otimes J_T\,)\,,$$

by collecting terms with the same matrices, we get

$\Psi = \sigma_\varepsilon^2\,(I_N \otimes E_T\,) + (\sigma_\varepsilon^2 + T\sigma_u^2)\,(I_N \otimes J_T\,) = \sigma_\varepsilon^2\,Q + \sigma_1^2\,P$, where , $\sigma_1^2 = (\sigma_\varepsilon^2 + T\sigma_u^2)$ and $\Psi^{-1} = \frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}$, $|\Psi| =$ product of its characteristic roots, [1] $\rightarrow |\Psi| = (\sigma_\varepsilon^2\,)^{N(T-1)}(\,\sigma_1^2\,)^N$.

Theorem1 : Let Y is $N_{NT}(\,F\theta\,,\Psi\,)$, then the likelihood estimators of parameters θ , σ_ε^2 , σ_u^2 are $\hat{\theta} = (\,F^T\,\Psi^{-1}\,F\,)^{-1}\,(\,F^T\,\Psi^{-1}\,Y\,)$, $\hat{\sigma}_\varepsilon^2 = \frac{1}{N\,(T-1\,)}\left(Y - F\,\hat{\theta}\right)^T Q\,(\left(Y - F\hat{\theta}\right)$ and

$\hat{\sigma}_u^2 = \frac{1}{N}\left(Y - F\,\hat{\theta}\right)^T P\left(Y - F\hat{\theta}\right) - \frac{1}{T}\hat{\sigma}_\varepsilon^2$.

Proof :

Since $Y \sim N_{NT}(\,F\theta\,,\Psi\,)$, then , the density function of Y is

$f(\,Y;\,\theta,\Psi\,) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp\{\frac{-1}{2}\,(\,Y - F\theta)^T\,\Psi^{-1}\,(Y - F\theta)\ \}$, then , the likelihood function is the joint density of the $Y's$ that is

$$L(Y\,;\theta,\Psi\,) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp\{\frac{-1}{2}\,(\,Y - F\theta)^T\,\Psi^{-1}\,(Y - F\theta)\ \},$$

$$\rightarrow L(Y;\,\theta,\sigma_\varepsilon^2\,,\sigma_1^2\,) = (2\pi)^{\frac{-NT}{2}}\,(\,\sigma_\varepsilon^2\,)^{\frac{-N(T-1)}{2}}\,(\sigma_1^2)^{\frac{-N}{2}}\,\exp\{\,\frac{-1}{2}\,(\,Y - F\theta)^T\,\Psi^{-1}\,(Y - F\theta)\ \},$$

$\therefore L(Y;\,\theta,\sigma_\varepsilon^2\,,\sigma_1^2\,) = (2\pi)^{\frac{-NT}{2}}\,(\,\sigma_\varepsilon^2\,)^{\frac{-N(T-1)}{2}}$

$$(\sigma_1^2)^{\frac{-N}{2}} \exp\{\frac{-1}{2}\left(Y - F\theta\right)^T \left[\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right]\left(Y - F\theta\right)\}, \quad (6)$$

Then,

$$\ln L = \frac{-NT}{2}\ln(2\pi) - \frac{N(T-1)}{2}\ln(\sigma_\varepsilon^2) - \frac{N}{2}\ln(\sigma_1^2) - \frac{1}{2}(Y - F\theta)^T\left[\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right](Y - F\theta).$$

Since $(Y - F\theta)^T \Psi^{-1}(Y - F\theta) = Y^T\Psi^{-1}Y - 2\theta^T F^T\Psi^{-1}Y + \theta^T F^T\Psi^{-1}F\theta$, then

$$\frac{\partial \ln L}{\partial \theta} = \frac{-1}{2}\left[-2F^T\Psi^{-1}Y + 2F^T\Psi^{-1}F\theta\right] = 0 \rightarrow F^T\Psi^{-1}F\theta = F^T\Psi^{-1}Y$$

$$\therefore \hat{\theta} = (F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}Y . \quad (7)$$

Now, derivative the $\ln L$ with respect to σ_ε^2 and σ_1^2 , we obtain

$$\frac{\partial \ln L}{\partial \sigma_\varepsilon^2} = \frac{-N(T-1)}{2\sigma_\varepsilon^2} + \frac{(Y - F\theta)^T Q (Y - F\theta)}{2\sigma_\varepsilon^4} \rightarrow \frac{-N\hat{\sigma}_\varepsilon^2(T-1) + (Y - F\hat{\theta})^T Q (Y - F\hat{\theta})}{2\hat{\sigma}_\varepsilon^4} = 0$$

$$\rightarrow (Y - F\hat{\theta})^T Q (Y - F\hat{\theta}) = N(T-1)\hat{\sigma}_\varepsilon^2$$

$$\therefore \hat{\sigma}_\varepsilon^2 = \frac{1}{N(T-1)}(Y - F\hat{\theta})^T Q (Y - F\hat{\theta}), \quad (8)$$

and, $\frac{\partial \ln L}{\partial \sigma_1^2} = \frac{-N}{2\sigma_1^2} - \frac{(Y - F\theta)^T P (Y - F\theta)}{2\sigma_1^4} = 0 \rightarrow N\hat{\sigma}_1^2 = (Y - F\hat{\theta})^T P (Y - F\hat{\theta})$.

$$\therefore \hat{\sigma}_1^2 = \frac{1}{N}(Y - F\hat{\theta})^T P (Y - F\hat{\theta}), \quad (9)$$

$$\rightarrow (\hat{\sigma}_\varepsilon^2 + T\hat{\sigma}_u^2) = \frac{1}{N}(Y - F\hat{\theta})^T P (Y - F\hat{\theta})$$

$$\therefore \hat{\sigma}_u^2 = \frac{1}{NT}(Y - F\hat{\theta})^T P (Y - F\hat{\theta}) - \frac{1}{T}\hat{\sigma}_\varepsilon^2 \quad \blacksquare$$

Theorem2: Let Y is $N_{NT}(F\theta, \Psi)$, then the maximum likelihood estimator of parameter θ is the best linear unbiased estimator (BLUE).

Proof:

Since $\hat{\theta} = (F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}Y \rightarrow E(\hat{\theta}) = E[(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}Y]$

$= (F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}E(Y) = [(F^T\Psi^{-1}F)^{-1}(F^T\Psi^{-1}F)]\theta = \theta.$

$Var(\hat{\theta}) = Var\,[\,(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}\,Y\,]$

$= (F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}\{Var\,(Y)\}\ \Psi^{-1}\,F\,(F^T\Psi^{-1}F)^{-1}$

$= (F^T\Psi^{-1}F)^{-1}\,(F^T\Psi^{-1}F)\,(F^T\Psi^{-1}F)^{-1} = (F^T\Psi^{-1}F)^{-1}.$

Now, le t $\hat{\theta}^{\bullet}$ = DY is another unbiased estimator for θ, where

$D = (F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1} + G$, where G is $(K+1)\times NT$ matrix,

Since $E(\hat{\theta}^{\bullet}) = \theta \rightarrow E(DY) = E\,\{[\,(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1} + G\,]\,Y\}$

$= [\,(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1} + G\,]\,E\,(Y) = [\,(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1} + G\,]\,F\theta$

$= (F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1}F\theta + G\,F\theta = I\,.\,\theta + 0 = \theta$, that is $G\,F = 0$

$Var(\hat{\theta}^{\bullet}) = Var\,(DY) = D\,Var\,Y\ D^T$

$= [\,(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1} + G\,]\,\Psi[\,(F^T\Psi^{-1}F)^{-1}F^T\Psi^{-1} + G\,]^T$

$= [\,(F^T\Psi^{-1}F)^{-1}\,(F^T\Psi^{-1}F)\,(F^T\Psi^{-1}F)^{-1} + G\,\Psi\,G^T = (F^T\Psi^{-1}F)^{-1} + G\,\Psi\,G^T$

$= Var(\hat{\theta}) + G\,\Psi\,G^T$, thus $Var\,(\hat{\theta}) < Var(\hat{\theta}^{\bullet})$ ■

Theorem3: suppose that Y is $N_{NT}(F\theta\,,\Psi)$,where F is $NT \times (K+1)$of rank$(K+1) < NT$ and $\theta = [\beta_0, \beta_1, \dots, \beta_K]^T$.Then the maximum likelihood estimator of θ is an efficient statistic for θ .

Proof:

Since $Y \sim N_{NT}(F\theta\,,\Psi)$, then , the density function of Y is

$f(Y;\ \theta, \Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp\{\frac{-1}{2}\,(Y-F\theta)^T\,\Psi^{-1}\,(Y-F\theta)\}$, and the likelihood function is

$L(Y\,;\theta,\Psi) = (2\pi)^{\frac{-NT}{2}}\,|\Psi|^{\frac{-1}{2}}\ \exp\{\frac{-1}{2}\,(Y-F\theta)^T\,\Psi^{-1}\,(Y-F\theta)\}$,

then,

$\ln L = \frac{-NT}{2}\ln(2\pi) - \frac{N(T-1)}{2}\ln(\sigma_\varepsilon^2) - \frac{N}{2}\ln(\sigma_1^2) - \frac{1}{2}\,(Y - F\theta)^T\left[\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right](Y - F\theta)$,

$\rightarrow \frac{\partial \ln L}{\partial\theta} = \frac{-1}{2}[-2F^T\Psi^{-1}Y + (F^T\Psi^{-1}F)\,\theta] \rightarrow \frac{\partial^2 \ln L}{\partial\theta^2} = -(F^T\,\Psi^{-1}F)^{-1}$.

Then , the Rao – Cramer lower bounded is

$$C.R.L = -E\left[\frac{\partial^2 \ln L}{\partial\theta^2}\right]^{-1} = (F^T\Psi^{-1}F)^{-1} = Var(\hat{\theta}) \quad \blacksquare$$

Theorem4: Let Y is $N_{NT}(F\theta, \Psi)$, then the maximum likelihood estimators of parameters θ , σ_ε^2 , and σ_1^2 are jointly sufficient for θ and σ_ε^2 , and σ_1^2 .

Proof:

Since $Y \sim N_{NT}(F\theta, \Psi)$, then , the density function of Y is

$f(Y;\ \theta, \Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp\{\frac{-1}{2}\ (Y - F\theta)^T\ \Psi^{-1}\ (Y - F\theta)\ \}$, we add and subtract $F\hat{\theta}$ to obtain

$$f(Y;\ \theta, \Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}} \exp[-\frac{1}{2}[(Y - F\hat{\theta} + F\hat{\theta} - F\theta)^T\Psi^{-1}(Y - F\hat{\theta} + F\hat{\theta} - F\theta)],$$

$$= (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}} exp\{-\frac{1}{2}\ [(Y - F\hat{\theta}) + F(\hat{\theta} - \theta)]^T\Psi^{-1}[(Y - F\hat{\theta}) + (\hat{\theta} - \theta)]\}.$$

$$\rightarrow f(Y;\theta,\Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}} exp[\frac{-1}{2}\{(Y - F\hat{\theta})^T\ \Psi^{-1}(Y - F\hat{\theta}) + (\hat{\theta} - \theta)^T F^T\ \Psi^{-1}F.\ (\hat{\theta} - \theta)\}].$$

$$f(Y;\ \theta,\Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp[-\frac{1}{2}\{(Y - F\hat{\theta})^T[\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}](Y - F\hat{\theta}) + (\hat{\theta} - \theta)^T F^T\Psi^{-1}F\ (\hat{\theta} - \theta)\}].$$

$$f(Y;\ \theta,\Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp[-\frac{1}{2}\{(Y - F\hat{\theta})^T\ \frac{Q}{\sigma_\varepsilon^2}(Y - F\hat{\theta}) + (Y - F\hat{\theta})^T\ \frac{P}{\sigma_1^2}(Y - F\hat{\theta}) + (\hat{\theta} - \theta)^T\ F^T\Psi^{-1}F\ (\hat{\theta} - \theta)\}]$$

$$f(Y;\ \theta,\Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}}\ \exp[-\frac{1}{2}\{(\frac{N(T-1)\hat{\sigma}_\varepsilon^2}{\sigma_\varepsilon^2} + \frac{N\hat{\sigma}_1^2}{\sigma_1^2}) + (\hat{\theta} - \theta)^T\ F^T\Psi^{-1}F\ (\hat{\theta} - \theta)\}].$$

We can now write the density as:

$f(Y;\ \theta,\Psi) =\ g\ (\hat{\theta}, \hat{\sigma}_\varepsilon^2, \hat{\sigma}_1^2, \theta, \sigma_\varepsilon^2, \sigma_1^2)\ h(Y)$, where h (Y) =1, therefore by the Neyman factorization theorem $\hat{\theta}$ and $\hat{\sigma}_\varepsilon^2, \hat{\sigma}_1^2$ are jointly sufficient for θ , σ_ε^2 and σ_1^2 $\blacksquare$

Theorem5: Let Y is $N_{NT}(F\theta, \Psi)$, the likelihood ratio test for $H_0: \theta = 0$ is $\frac{(Y^T Y)^{-\frac{NT}{2}}}{[(Y - F\hat{\theta})^T (Y - F\hat{\theta})]^{\frac{-NT}{2}}} \sim F(NT, NT - K - 1)$

Proof:

Since the likelihood estimators for $\theta, \sigma_\varepsilon^2$ and σ_1^2 are

$\hat{\theta} = (F^T \Psi^{-1} F)^{-1} F^T \Psi^{-1} Y$, $\hat{\sigma}_\varepsilon^2 = \frac{1}{N(T-1)} (Y - F\hat{\theta})^T Q (Y - F\hat{\theta})$ and

$\hat{\sigma}_1^2 = \frac{1}{N} (Y - F\hat{\theta})^T P (Y - F\hat{\theta})$, and, the maximum likelihood function is

$L = (2\pi)^{\frac{-NT}{2}} (\sigma_\varepsilon^2)^{\frac{-N(T-1)}{2}} (\sigma_1^2)^{\frac{-N}{2}} \exp\{\frac{-1}{2} (Y - F\theta)^T \left(\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right) (Y - F\theta)\}$, then, the likelihood function under H_1, $(\max_{H_1} L(\theta, \sigma_\varepsilon^2, \sigma_1^2))$ is

$$L_1 = (2\pi)^{\frac{-NT}{2}} [\frac{1}{N(T-1)} (Y - F\hat{\theta})^T Q (Y - F\hat{\theta})]^{\frac{-N(T-1)}{2}} [\frac{1}{N} (Y - F\hat{\theta})^T P (Y - F\hat{\theta})]^{\frac{-N}{2}}$$

$$\exp\left\{-\frac{1}{2}\left[\frac{(Y - F\hat{\theta})^T Q (Y - F\hat{\theta})}{\frac{1}{N(T-1)}(Y - F\hat{\theta})^T Q (Y - F\hat{\theta})} + \frac{(Y - F\hat{\theta})^T P (Y - F\hat{\theta})}{\frac{1}{N}(Y - F\hat{\theta})^T P (Y - F\hat{\theta})}\right]\right\}$$

$$L_1 = (2\pi)^{\frac{-NT}{2}} [\frac{1}{N(T-1)} (Y - F\hat{\theta})^T Q (Y - F\hat{\theta})]^{\frac{-N(T-1)}{2}} [\frac{1}{N} (Y - F\hat{\theta})^T P (Y - F\hat{\theta})]^{\frac{-N}{2}} e^{-\frac{NT}{2}}.$$

And the likelihood function under H_0, $\max_{H_0} L(\theta, \sigma_\varepsilon^2, \sigma_1^2) = \text{Max}\, L(0, \sigma_\varepsilon^2, \sigma_1^2)$

$$\rightarrow L_0(0, \sigma_\varepsilon^2, \sigma_1^2) = (2\pi)^{\frac{-NT}{2}} (\sigma_\varepsilon^2)^{\frac{-N(T-1)}{2}} (\sigma_1^2)^{\frac{-N}{2}} \exp\left[\frac{-1}{2} Y^T \left(\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right) Y\right] \rightarrow$$

$$\ln L_0(0, \sigma_\varepsilon^2, \sigma_1^2) = -\frac{NT}{2}\ln(2\pi) - \frac{N(T-1)}{2}\ln \sigma_\varepsilon^2 - \frac{N}{2}\ln \sigma_1^2 - \frac{1}{2} Y^T \left(\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right) Y,$$

$$\rightarrow \frac{\partial \ln L}{\partial \sigma_\varepsilon^2} = \frac{-N(T-1)}{2\sigma_\varepsilon^2} + \frac{Y^T Q Y}{2\sigma_\varepsilon^4} = 0 \rightarrow \frac{-N(T-1)\sigma_\varepsilon^2 + Y^T Q Y}{2\sigma_\varepsilon^4} = 0 \rightarrow Y^T Q Y = N(T-1)\hat{\sigma}_\varepsilon^2$$

$$\therefore \hat{\sigma}_\varepsilon^2 = \frac{1}{N(T-1)} Y^T Q Y,$$

and $\frac{\partial \ln L}{\partial \sigma_1^2} = \frac{-N}{2\sigma_1^2} + \frac{Y^T P Y}{2\sigma_1^4} = 0 \rightarrow \frac{-N\sigma_1^2 + Y^T P Y}{2\sigma_1^4} = 0 \rightarrow N\hat{\sigma}_1^2 = Y^T P Y \rightarrow \hat{\sigma}_1^2 = \frac{1}{N} Y^T P Y.$

Then, the likelihood function under H_0 is

$$\max_{H_0} L(\theta, \sigma_\varepsilon^2, \sigma_1^2) = \text{Max}\, L(0, \sigma_\varepsilon^2, \sigma_1^2)$$

$$= (2\pi)^{\frac{-NT}{2}} \left[\frac{1}{N(T-1)} Y^T Q Y\right]^{\frac{-N(T-1)}{2}} \left[\frac{1}{N} Y^T P Y\right]^{\frac{-N}{2}}$$

$$= (2\pi)^{\frac{-NT}{2}} \left[\frac{1}{N(T-1)} Y^T Q Y\right]^{\frac{-N(T-1)}{2}} \left[\frac{1}{N} Y^T P Y\right]^{\frac{-N}{2}} e^{-\frac{NT}{2}}.$$

Then , the likelihood ratio is

$$\lambda = \frac{\text{Max}H_0}{\text{Max}H_1} = \frac{(2\pi)^{\frac{-NT}{2}} \left[\frac{1}{N(T-1)} Y^T Q Y\right]^{\frac{-N(T-1)}{2}} \left[\frac{1}{N} Y^T P Y\right]^{\frac{-N}{2}} e^{-\frac{NT}{2}}}{(2\pi)^{\frac{-NT}{2}} \left[\frac{1}{N(T-1)} (Y - F\hat{\theta})^T Q (Y - F\hat{\theta})\right]^{\frac{-N(T-1)}{2}} \left[\frac{1}{N} (Y - F\hat{\theta})^T P (Y - F\hat{\theta})\right]^{\frac{-N}{2}} e^{-\frac{NT}{2}}}$$

$$= \frac{(Y^T Y)^{-\frac{NT}{2}}}{[(Y - F\hat{\theta})^T (Y - F\hat{\theta})]^{\frac{-NT}{2}}} = \frac{\frac{\hat{\omega}_0^T \hat{\omega}_0}{NT}}{\frac{\hat{\omega}_1^T \hat{\omega}_1}{NT-k-1}} \sim F(NT, NT - K - 1). \quad (10)$$

Where, $\hat{\omega}_1^T \ \hat{\omega}_1 = (Y - F\hat{\theta})^T (Y - F\hat{\theta})$ is $\chi^2(NT - K - 1)$ and $\hat{\omega}_0^T \ \hat{\omega}_0 = (Y^T Y)$ is $\chi^2(NT)$. ■

Theorem6: Let Y is $N_{NT}(F\theta, \Psi)$, the likelihood ratio test for $H_0 : A\theta = 0$, where A is $q \times (K+1)$ matrix of constants is

$$\left[\frac{\frac{\hat{\omega}_0^T \hat{\omega}_0 + \hat{\theta}^T A^T \left[A (F^T F)^{-1} A^T\right]^{-1} A\hat{\theta}}{q}}{\frac{\hat{\omega}_1^T \hat{\omega}_1}{NT-K-1}}\right]^{-\frac{NT}{2}} \sim F(q, NT - K - 1)$$

Proof:

The likelihood function is

$L = (2\pi)^{\frac{-NT}{2}} (\sigma_\varepsilon^2)^{\frac{-N(T-1)}{2}} (\sigma_1^2)^{\frac{-N}{2}} \exp\{\frac{-1}{2} (Y - F\theta)^T \left(\frac{Q}{\sigma_\varepsilon^2} + \frac{P}{\sigma_1^2}\right) (Y - F\theta)\}$, then for test $H_0 : A\theta = 0$. Let $L^\bullet = \ln L + \lambda^T A\, \theta$

$$= -\frac{NT}{2} \ln(2\pi) - \frac{N(T-1)}{2} \ln(\sigma_\varepsilon^2) - \frac{N}{2} \ln(\sigma_1^2) - \frac{1}{2} (Y - F\theta)^T$$

$$\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)(Y-F\theta)+\lambda^T A\,\theta$$

$$\rightarrow \frac{\partial L^{\bullet}}{\partial \theta}=2\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)Y-2\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\theta+A^T\lambda=0\,, \tag{11}$$

$$\rightarrow \frac{\partial L^{*}}{\partial \lambda}=A\,\theta=0\,, \tag{12}$$

$$\rightarrow \frac{\partial \ln L^{\bullet}}{\partial \sigma_\varepsilon^2}=\frac{-N(T-1)}{2\sigma_\varepsilon^2}+\frac{(Y-F\hat{\theta})^T Q\,(Y-F\hat{\theta})}{2\sigma_\varepsilon^4}=0\,, \tag{13}$$

$$\rightarrow \frac{\partial \ln L^{\bullet}}{\partial \sigma_1^2}=\frac{-N}{2\sigma_1^2}-\frac{(Y-F\hat{\theta})^T\,P\,(Y-F\hat{\theta})}{2\sigma_1^4}=0\,. \tag{14}$$

Multiply (11) by $A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}$ to obtain

$$2A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)Y-\ 2A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}[\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]\theta$$

$$+\ A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}A^T\lambda=\ 0$$

$\rightarrow 2A\,\hat{\theta}-2A\theta+\ A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}A^T\lambda=0,$ since $A\theta=0$

$$\rightarrow\ A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}A^T\lambda=2A\hat{\theta}\ \ \rightarrow \lambda=-2\,(\,A\ [\,F^T\left(\frac{Q}{\sigma_\varepsilon^2}+\frac{P}{\sigma_1^2}\right)F\,]^{-1}A^T\,)^{-1}\,A\hat{\theta}.$$

From (13) and (14) ,we get

$$\hat{\sigma}_\varepsilon^2=\frac{1}{N\,(T-1\,)}(Y-F\hat{\theta})^T Q\,(\,Y-F\hat{\theta})\,, \tag{15}$$

$$\hat{\sigma}_1^2=\frac{1}{N}(Y-F\hat{\theta})^T P(\,Y-F\hat{\theta})\,. \tag{16}$$

Then ,

$$\lambda=-2\left(A\ [\ F^T\left(\frac{Q}{\frac{1}{N\,(T-1\,)}(Y-F\hat{\theta})^T Q\,(\,Y-F\hat{\theta})}+\frac{P}{\frac{1}{N}(Y-F\hat{\theta}\,)^T\,P\,(\,Y-F\hat{\theta})}\right)F\,]^{-1}A^T\right)^{-1}\ A\hat{\theta}$$

$$\rightarrow \lambda=-2\left(A\left[\frac{N(T-1)F^TF+NF^TF}{(Y-F\hat{\theta})^T(Y-F\hat{\theta})}\right]^{-1}A^T\right)A\hat{\theta}\rightarrow\ \lambda=-2\left(A\left[\frac{NT\,F^TF}{(Y-F\hat{\theta})^T(Y-F\hat{\theta})}\right]^{-1}A^T\right)A\hat{\theta}.$$

Now, substitute λ in (11) to obtain

$$2\,F^T\left(\frac{Q}{\frac{1}{N\,(T-1)}(Y-F\hat{\theta})^T Q\,(Y-F\hat{\theta})}+\frac{P}{\frac{1}{N}(Y-F\hat{\theta})^T P\,(Y-F\hat{\theta})}\right)Y \qquad -2\,F^T\left(\frac{Q}{\frac{1}{N\,(T-1)}(Y-F\hat{\theta})^T Q\,(Y-F\hat{\theta})}+\frac{P}{\frac{1}{N\,(T-1)}(Y-F\hat{\theta})^T Q\,(Y-F\hat{\theta})}\right)F\theta-2\left(A\left[\frac{NT\,F^TF}{(Y-F\hat{\theta})^T\,(Y-F\hat{\theta})}\right]^{-1}A^T\right)^{-1}A\hat{\theta}=0$$

$$\rightarrow \frac{2NT}{(Y-F\hat{\theta})^T\,(Y-F\hat{\theta})}\,[\,F^TY-F^TF\theta-A^T[\,A\,(F^TF)^{-1}A^T\,]^{-1}\,A\hat{\theta}=0$$

$$\rightarrow F^TY-F^TF\theta-A^T[\,A\,(F^TF)^{-1}A^T\,]^{-1}\,A\hat{\theta}=0$$

$$\hat{\theta}_0=(F^TF)^{-1}\left(F^TY-A^T[\,A\,(F^TF)^{-1}A^T\,]^{-1}\,A\hat{\theta}\right). \qquad (17)$$

Then ,

$$Y-F\hat{\theta}_0=Y-F\hat{\theta}-F\,(F^TF\,)^{-1}\,A^T[\,A\,(F^TF\,)^{-1}A^T\,]^{-1}A\hat{\theta},$$

$$u_0=\widehat{\omega}-F(F^TF)^{-1}A^T[\,A\,(F^TF\,)^{-1}A^T\,]^{-1}A\hat{\theta},$$

Where u_0 is estimated residual from the restricted model

$$u_0^Tu_0=\left\{\widehat{\omega}^T-\hat{\theta}^TA^T[A\,(F^TF)^{-1}A^T]^{-1}A(F^TF)^{-1}F^T\right\}\{\widehat{\omega}-F(F^TF)^{-1}A^T\,[A\,(F^TF)^{-1}A^T]^{-1}A\hat{\theta}\}$$

$$\rightarrow u_0^Tu_0=\ \widehat{\omega}^T\widehat{\omega}-\widehat{\omega}^TF\,A(F^TF)^{-1}[A\,(F^TF)^{-1}\,A^T]^{-1}A\hat{\theta}-\hat{\theta}^TA^T\,[A\,(F^TF)^{-1}\,A^T]^{-1}$$
$$A(F^TF)^{-1}F^T\,\widehat{\omega}+A^T\hat{\theta}^T\,[A\,(F^TF)^{-1}\,A^T]^{-1}A(F^TF)^{-1}F^TF\,A\,(F^TF)^{-1}$$
$$[A\,(F^TF)^{-1}\,A^T]^{-1}A\hat{\theta}\rightarrow u_0^Tu_0=\widehat{\omega}^T\widehat{\omega}+A^T\,\hat{\theta}^T[A\,(F^TF)^{-1}\,A^T]^{-1}A\hat{\theta}.$$

We can note that $\widehat{\omega}^TF=\left(Y-F\hat{\theta}\right)^TF=\left(Y^T-\hat{\theta}^TF^T\right)F$

$$=Y^TF-(Y^T\Psi^{-1}F(F^T\Psi^{-1}F)^{-1}F^T)F$$

$$=Y^TF-Y^TF=0\ ,$$

In the same way we can prove that $F^T\widehat{\omega}=0$

$\rightarrow u_0^T u_0 - \hat{\omega}^T\hat{\omega} = A^T\hat{\theta}^T\,[A\,(F^TF)^{-1}\,A^T]^{-1}A\hat{\theta}$

$$\rightarrow \frac{\frac{u_0^T u_0 - \hat{\omega}^T\hat{\omega}}{q}}{\frac{\hat{\omega}^T\hat{\omega}}{NT-K-1}} \sim F(\,q\,, NT-K-1).$$

The likelihood function under H_0 is

$$Max_{H_0} L\,(\,\theta\,,\sigma_\varepsilon^2\,,\sigma_1^2\,) = L\left(\,\hat{\theta}_0\,,\hat{\sigma}^2_{\varepsilon(0)},\hat{\sigma}^2_{\varepsilon(0)}\right)$$

$$=(2\pi)^{\frac{-NT}{2}}[\frac{1}{N\,(T-1\,)}(Y-F\hat{\theta}_0)^TQ\left(\,Y-F\hat{\theta}_0\right)]^{\frac{-N(T-1)}{2}}\,[\frac{1}{N}(Y-\,F\hat{\theta}_0)^TP\left(\,Y-F\hat{\theta}_0\right)]^{\frac{-N}{2}}\;e^{-\frac{NT}{2}},$$

and the likelihood function under H_1 is

$$Max_{H_1}\,L\,(\,\theta\,,\sigma_\varepsilon^2\,,\sigma_1^2\,) = L\left(\hat{\theta}\,,\hat{\sigma}_\varepsilon^2,\hat{\sigma}_1^2\right)$$

$$=\,(2\pi)^{\frac{-NT}{2}}\,[\frac{1}{N\,(T-1\,)}(Y-F\hat{\theta})^TQ\left(\,Y-F\hat{\theta}\right)]^{\frac{-N(T-1)}{2}}\,[\frac{1}{N}(Y-\,F\hat{\theta}\,)^TP\left(\,Y-\,F\hat{\theta}\right)]^{\frac{-N}{2}}\;e^{-\frac{NT}{2}}.$$

Thus, the likelihood ratio is

$$\lambda = \frac{Max_{H_0}}{Max_{H_1}} = \frac{(2\pi)^{\frac{-NT}{2}}\,[\frac{1}{N\,(T-1\,)}(Y-F\,\hat{\theta}_0)^TQ\left(\,Y-F\hat{\theta}_0\right)]^{\frac{-N(T-1)}{2}}\,[\frac{1}{N}(Y-F\hat{\theta}_0)^TP\left(\,Y-F\hat{\theta}_0\right)]^{\frac{-N}{2}}\;e^{-\frac{NT}{2}}}{(2\pi)^{\frac{-NT}{2}}\,[\frac{1}{N\,(T-1\,)}(Y-F\,\hat{\theta})^TQ\left(\,Y-F\,\hat{\theta}\right)]^{\frac{-N(T-1)}{2}}\,[\frac{1}{N}(Y-F\hat{\theta})^TP\left(\,Y-F\hat{\theta}\right)]^{\frac{-N}{2}}\;e^{-\frac{NT}{2}}}$$

$$= \frac{[\,(Y-F\hat{\theta}_0)^T\left(\,Y-F\hat{\theta}_0\right)]^{-\frac{NT}{2}}}{[\,(Y-F\hat{\theta})^T\left(\,Y-F\hat{\theta}\right)]^{-\frac{NT}{2}}} = \left[\frac{\frac{\hat{\omega}^T\hat{\omega}+\hat{\theta}^TA^T\left[A\left(F^TF\right)^{-1}A^T\right]^{-1}A\hat{\theta}}{q}}{\frac{\hat{\omega}^T\hat{\omega}}{NT-K-1}}\right]^{-\frac{NT}{2}} \qquad \blacksquare$$

3. CONCLUSION

The conclusions which are obtained throughout this paper are given as follows:

1. The maximum likelihood estimators of parameters θ , σ_ε^2 , σ_u^2 of panel data model are

$\hat{\theta} = (\,F^T\,\Psi^{-1}\,F\,)^{-1}\,(\,F\,\Psi^{-1}\,Y\,)\,,\,\hat{\sigma}_\varepsilon^2 = \frac{1}{N\,(T-1\,)}\left(Y-F\,\hat{\theta}\right)^T Q\,(\,\left(Y-F\hat{\theta}\right)$ and

$\hat{\sigma}_u^2 = \frac{1}{N}\left(Y-F\,\hat{\theta}\right)^T P\left(Y-F\hat{\theta}\right) - \frac{1}{T}\hat{\sigma}_\varepsilon^2.$

2. The maximum likelihood estimator of parameter θ of panel data model is the best linear unbiased estimator (BLUE).
3. The maximum likelihood estimator of parameter θ of panel data model is an efficient statistic for θ.
4. The maximum likelihood estimators of parameters θ, σ_ε^2 , and σ_1^2 of panel data model are jointly sufficient for θ and σ_ε^2 , and σ_1^2 .
5. The likelihood ratio test for $H_0 : \theta = 0$ in the panel data model is

$$\frac{(Y^T Y)^{-\frac{NT}{2}}}{[(Y - F\hat{\theta})^T (Y - F\hat{\theta})]^{\frac{-NT}{2}}} \sim F(NT, NT - K - 1).$$

6. The likelihood ratio test for $H_0 : A\theta = 0$,where A is $q \times (K+1)$ matrix of constants in the panel data model is

$$\left[\frac{\frac{\hat{\omega}^T\hat{\omega}+\hat{\theta}^T A^T\left[A\left(F^T F\right)^{-1}A^T\right]^{-1}A\hat{\theta}}{q}}{\frac{\hat{\omega}^T\hat{\omega}}{NT-K-1}}\right]^{-\frac{NT}{2}} \sim F(q, NT-K-1).$$

4. REFERENCES

1. Baltagi, badi,” Econometric Analysis of panel data “, John Wily & Sons Inc. third edition,(2005).
2. Graybill, F.A. “Theory and Application of the Linear Model “North Scituate, MA; Duxbury Press,(1976).
3. Husio, Cheng, “Analysis of panel data “, Second edition, Cambridge university Press, (2003).
4. Hogg, Robert V., McKean, Joseph W.& Craig, Allen T., “ Introduction to Mathematical Statistics “, seven edition ,Pearson Education , Inc., (2013).
5. Mood, Alexander M., Graybill, Franklin A. & Boes , Duane C., “ Introduction to the theory of statistics “, third edition. McGraw- Hill, Inc., (1974).
6. Rencher , Alvin C. and Schaalje, G. Bruce, “ Linear Models in Statistics “ ,second edition, Wiley-Interscience, A John Wiley & Sons ,Inc., Publication,(2008).
7. Searal ,S.R.,” Linear Model .New York: Wiley, (1971).
8. Seber , G .A .F. and A.J. Lee, “ Linear Regression Analysis (2nd ed.). Hoboken, NJ: Wiley,(2003).
9. Sun, Jianguo & Zhao, Xingqiu , “ Statistical Analysis of Panel Count Data “, Springer , (2013).

10. Verbeke,Greet & Molenberghs, Greet , " Linear Mixed Model for Longitudinal Data" with 128 Illustrations ,Springer, (2000) .